AlphaStar Math Courses

Curriculum

AlphaStar
Academy

Levels Overview

AlphaStar Math Program has eight levels with each level having four subject offerings: Algebra, Counting, Geometry, and Number Theory. Each of these subject courses are taught in twelve chapters. For each chapter, there are Lecture Notes, Example Problems, Exercise (HW) Problems, and a Quiz.

Below is a short description for each level. For more information about our Math Courses and diagnostic exams for each level, please visit AlphaStar Academy website.

In this document, you can find objectives along with a sample problem for each chapter of our courses. To go directly to the relevant course, you can click on the course name from the Table of Contents page.

MC15 (Pre-MathCounts)

For students in grades 4-6 who are good at school math.
Target Contests: MOEMS, Math League, Math Kangaroo, Noetic, Alpha Math Contest

MC20 (AMC 8/MathCounts Basic)

For middle school students who are familiar with Pre-Algebra topics and want to start training for middle school math competitions.
Target Contests: AMC 8, MathCounts

MC25 (AMC 8/MathCounts Advanced)

For middle school students who are familiar with Algebra-1 topics and want to master middle school math competitions
Target Contests: AMC 8, MathCounts, Berkeley mini Math Tournament

MC30 (AMC 10/12 Basic)

For middle and high school students who are familiar with Algebra-1 topics and want to start training for high school math competitions AMC 10/12
Target Contests: AMC 10, AMC 12

MC35 (AMC 10/12 Advanced)

For middle and high school students who are very comfortable with Algebra-1 topics and want to master high school math competitions AMC 10/12
Target Contests: AMC 10/12, Harvard MIT Math Tournament November

MC40 (AIME Basic)

For students who are comfortable with AIME qualification and want to start training for AIME, ARML, and high school competitions organized by prestigious universities.
Target Contests: AIME, ARML, Harvard MIT Math Tournament, Princeton Math Competition, Caltech Harvey Mudd Math Competition, Stanford Math Tournament, Berkeley Math Tournament

MC45 (AIME Advanced)

For students who want to master AIME, ARML, and high school competitions organized by prestigious universities
Target Contests: AIME, ARML, Harvard MIT Math Tournament (HMMT), Caltech Harvey Mudd Math Competition (CHMMC), Stanford Math Tournament (SUMO)

MC50 (USA(J)MO)

An introductory proof writing course. For students who are confident about USAJMO/USAMO qualification and are willing to work one hour on a single math Olympiad problem.
Target Contests: USAJMO, USAMO

Contents

MC15A
Pre-MathCounts Algebra

Chapter 1: Integers & Arithmetic

- Order of Operations with Integers and four basic operations
- Word problems only using arithmetic with integers without the need of unknowns

Sample Problem:
(AMC8-2014-5) Margie's car can go 32 miles on a gallon of gas, and gas currently costs \$4 per gallon. How many miles can Margie drive on \$20 worth of gas?

(A) 64 **(B)** 128 **(C)** 160 **(D)** 320 **(E)** 640

Chapter 2: Rate & Proportion

- Definition of rate and proportion
- Given the ratio of two numbers and one of the numbers, finding the other one
- Given the ratio of two numbers and their sum (or difference), finding the numbers

Sample Problem:
(AMC10-2006-A3) The ratio of Mary's age to Alice's age is $3 : 5$. Alice is 30 years old. How many years old is Mary?

(A) 15 **(B)** 18 **(C)** 20 **(D)** 24 **(E)** 50

Chapter 3: Fractions

- Different types of fractions (proper, improper, mixed, common)
- Basic arithmetic with fractions and related word problems

Sample Problem:
(PiMC-2016-Individual-5) Simplify

$$8 \times \left(\frac{2}{6} + \frac{3}{24} + \frac{1}{24}\right).$$

Chapter 4: Decimals

- Arithmetic and applications of decimals and related word problems
- Conversions between fractions and decimals

Sample Problem:
(Jocelyn Zhu) One ping is 0.4 of a pong. One pong is 0.6 of a pang. How many pangs is a ping? Express your answer as a decimal to the nearest hundredth.

Chapter 5: Percent

- Conversions between percent and fractions/decimals
- Word problems involving percent (tax, tip, interest, etc.)

Sample Problem:
(AMC8-2004-6) After Sally takes 20 shots, she has made 55% of her shots. After she takes 5 more shots, she raises her percentage to 56%. How many of the last 5 shots did she make?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

Chapter 6: Exponents

- Basic properties of exponents (multiplying, dividing, raising an exponent to another exponent)
- Perfect powers (squares and cubes) and square roots (of perfect squares)

Sample Problem:
(Hope Chen) If A and B are integers, $A^B = 27$, and $B \geq A$, What is A^5?

Chapter 7: Word Problems

- Converting word problems into mathematical equations
- Solving one-unknown linear equations

Sample Problem:
(PiMC-2016-Individual-13) Danielle and Rachel have an equal number of berries. After Danielle eats two of her own berries, Celine comes and takes two-thirds of Danielle's remaining berries, and Rachel eats 80% of her own berries. At the end, Danielle and Rachel have the same number of berries again. How many berries did Danielle have at the beginning?

Chapter 8: Time & Travel

- Unit conversions (ounces to pounds, square feet to square yards, etc.)
- Distance = Rate $\times$ Time
- Average speed, relative speed

Sample Problem:
(AMC10-2008-A1) A bakery owner turns on his doughnut machine at 8:30 AM. At 11:10 AM the machine has completed one third of the day's job. At what time will the doughnut machine complete the job?

(A) 1:50 PM **(B)** 3:00 PM **(C)** 3:30 PM **(D)** 4:30 PM **(E)** 5:50 PM

Chapter 9: Sequences-1

- Definition of a sequence
- Mean, median, mode, range

Sample Problem:
(AMC8-2004-9) The average of the five numbers in a list is 54. The average of the first two numbers is 48. What is the average of the last three numbers?

(A) 55 **(B)** 56 **(C)** 57 **(D)** 58 **(E)** 59

Chapter 10: Sequences-2

- Arithmetic sequences
- Geometric sequences
- Recursive sequences

Sample Problem:
(CEMC-2016-Gauss7-15) Sophia did push-ups every day for 7 days. Each day after the first day, she did 5 more push-ups than the day before. In total she did 175 push-ups. How many push-ups did Sophia do on the last day?

(A) 55 **(B)** 35 **(C)** 50 **(D)** 45 **(E)** 40

Chapter 11: Work

- Word problems involving rates of work/output
- Similarity between work and $D = R \times T$

Sample Problem:
(PiMC-2016-Team-10) Christie is making paper cranes while Katrina unfolds them. It takes Christie 2 minutes to make one paper crane and it takes Katrina 15 seconds to unfold one. Christie starts with 21 cranes and makes more while Katrina unfolds them. How many seconds will pass before Christie has no cranes left?

Chapter 12: Functions & Operations

- Informal treatment, definition of a function, examples
- Absolute value, floor/ceiling value
- Operators

Sample Problem:
(Richard Spence) If $a \spadesuit b = |a+b|$, then what are all possible values of x, if $x \spadesuit 20 = 19$?

MC15C
Pre-MathCounts Counting

Chapter 1: Addition Principle

- Counting the number of elements in a set or sequence quickly (e.g. how many multiples of 5 are between 100 and 1000 inclusive?)
- Using addition to count the number of ways to accomplish a task

Sample Problem:
(Hope Chen) How many numbers between 1 and 32 (inclusive) are divisible by 3?

Chapter 2: Multiplication Principle

- Using multiplication to count the number of ways to accomplish a task

Sample Problem:
(Richard Spence) In poker, a five-card hand is called a *four-of-a-kind* if there are four cards of the same rank, and a fifth card of different rank. An example four-of-a-kind is A♣A♠A♢A♡4♠. How many different four-of-a-kind hands are there?

Chapter 3: Permutations

- Definition of the factorial (n!)
- Permutations - Finding the number of ways to choose k items from a set of n items where the ordering of items is important

Sample Problem:
(Rohan Cherukuri) Eight students have to get into line. How many ways can they do this, if the oldest one must be at the front?

Chapter 4: Combinations

- How to compute n choose k
- Difference between combinations and permutations taken k at a time
- Number of ways to form a committee using combinations

Sample Problem:
(Jocelyn Zhu) Victor has 6 friends. He has a movie pass that allows him to bring 3 more friends for free. How many different groups of three friends can Victor choose to go to the movies?

Chapter 5: Casework

- Using casework to solve a variety of counting problems

Sample Problem:
(CEMC-2006-Gauss7-24) A triangle can be formed having side lengths 4, 5 and 8. It is impossible, however, to construct a triangle with side lengths 4, 5 and 10. Using the side lengths 2, 3, 5, 7 and 11, how many different triangles *with exactly two equal sides* can be formed?

(A) 8 **(B)** 5 **(C)** 20 **(D)** 10 **(E)** 14

Chapter 6: Complementary Counting

- Using complementary counting - we count the number of ways we don't want first, then subtract this result from the total

Sample Problem:
(Rohan Cherukuri) A school needs to select a debate team of size 3 from its pool of 10 people. Abraham and Stephen are both in this pool. In how many ways can at least one of them be selected for the 3 person team?

Chapter 7: Overcounting

- Using the technique of overcounting by counting more than what we need, then subtracting or dividing to account for the extra cases considered

Sample Problem:
(Jocelyn Zhu) How many different ways can you arrange the letters in the word "ALPHA"?

Chapter 8: Counting Sets

- Simple definition of a set in mathematics (finite sets, including the empty set)
- Union and intersection of two sets
- Venn diagrams, Principle of Inclusion-Exclusion, other counting problems involving sets

Sample Problem:
(MathCounts-2014-Chapter-Countdown-54) Among the 65 cheerleaders at an Austin middle school, 25 were Cowboys fans, 42 were Texans fans, and 6 were not fans of either team. How many cheerleaders were fans of both teams?

Chapter 9: Counting Shapes & Paths

- Counting the number of shapes or paths systematically (e.g. using combinations)

Sample Problem:
(Abby Berry) A class wants to walk to Walmart to buy some Jolly Ranchers. Imagine that the path they are walking is along a grid. They are starting at (0,0) and Walmart is at (3,4), and they can only walk straight or go right. Also, they may only turn at lattice points. How many ways are there for them to get to Walmart?

Chapter 10: Counting with Digits

- Counting problems involving digits of a number

- Palindromic numbers

Sample Problem:
(Rohan Cherukuri) How many 6 digit palindromes have a 0 in them?

Chapter 11: Probability-1

- Definition of probability (number of desired outcomes divided by the total number of outcomes)
- Solving probability problems using other counting techniques (e.g. complementary counting)

Sample Problem:
(MathCounts-2012-School-Sprint-11) If you toss two standard six-sided dice, what is the probability that you will get a 3 on at least one die? Express your answer as a common fraction.

Chapter 12: Probability-2

- Independent versus dependent events

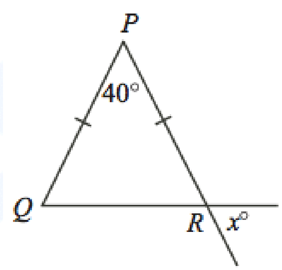
Sample Problem:
(Richard Spence) A standard 52-card deck is shuffled, and the top card is drawn. This card is shuffled back into the deck, and a second card is drawn. What is the probability that the first card is an Ace and the second card is a spade?

MC15G
Pre-MathCounts Geometry

Chapter 1: Angles-1

- Angles and terminology
- Parallel, perpendicular, and transversal lines
- Sum of the degree measures in a triangle
- Types of triangles and their properties

Sample Problem:
(CEMC-2008-Gauss7-13) In the diagram, $\triangle PQR$ is isosceles. The value of x is



(A) 40 **(B)** 70 **(C)** 60 **(D)** 30 **(E)** 110

Chapter 2: Angles-2

- Different types and names of polygons (by number of side lengths; regular polygons)

- Convex and concave polygons
- Sum of internal and external angles in an n-sided polygon
- Diagonals of a polygon

Sample Problem:
(Richard Spence) The interior angles of a convex pentagon form an arithmetic sequence. If the smallest angle measures 64°, what is the measure of the second-smallest angle?

Chapter 3: Angles-3

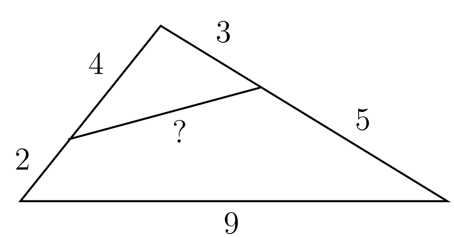
- Properties of inscribed angles in circles
- Major/minor arcs, sectors, tangent lines to circles

Sample Problem:
(Richard Spence) Points A and B are on circle O such that $\angle AOB = 48°$. Point P is on major arc AB such that $\triangle APB$ is isosceles. What is the degree measure of $\angle OAP$?

Chapter 4: Similarity

- Definition of similarity, congruence, stretch factor
- Congruence/similarity axioms (SSS, SAS, ASA, AA)
- Ratios of lengths between similar triangles

Sample Problem:
(Richard Spence) In the figure below, what is the length of the missing segment?



Chapter 5: Length-1

- Perimeter of polygons
- Triangle inequality

Sample Problem:
(Jocelyn Zhu) A triangle has perimeter 14. What is the largest possible integer side length of the triangle?

Chapter 6: Length-2

- Definition of legs, hypotenuse of a right triangle
- Pythagorean theorem
- Special right triangles (30-60-90 and 45-45-90 triangles)
- Pythagorean triples

Sample Problem:
(Hope Chen) What is the perimeter of a right triangle with legs 15 and 20?

Chapter 7: Length-3

- Circumference of a circle
- Chords, power of a point

Sample Problem:
(Hope Chen) There are two chords BC and DE in a circle that intersect at point A. If $AB = 1$, $AC = 8$, and $AE = 2$, what is the length of chord DE?

Chapter 8: Area-1

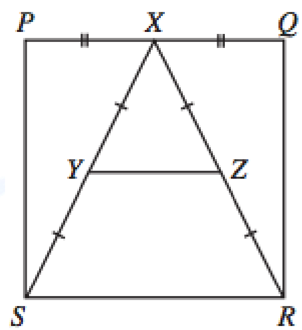
- Definition of area and square units
- Area formulas for squares, rectangles, parallelograms, rhombuses

Sample Problem:
(Hope Chen) The area of a rectangle is 27. If the length is three times the length of the width, What is the length of the rectangle?

Chapter 9: Area-2

- Area of a triangle (base × height/2)
- Area formula for a trapezoid
- Similar triangles and areas

Sample Problem:
(CEMC-2002-Gauss7-24) $PQRS$ is a square with side length 8. X is the midpoint of side PQ, and Y and Z are the midpoints of XS and XR, respectively, as shown. The area of trapezoid $YZRS$ is



(A) 24 **(B)** 16 **(C)** 20 **(D)** 28 **(E)** 32

Chapter 10: Area-3

- Area of a circle
- Area of more complex shapes
- Review of area formulas

Sample Problem:
(Richard Spence) A circle is inscribed inside a square of side length 8. What is the total area of the region inside the square but outside the circle?

Chapter 11: Analytic Geometry

- Cartesian coordinate system, graphing points
- Definition of the slope of a line
- Slope-intercept form, point-slope form
- Midpoint and distance formulas

Sample Problem:
(Jennifer Zhu) What is the y-coordinate of the y-intercept of the line that passes through $(1, 5)$ and $(3, 9)$?

Chapter 12: 3D

- Applications of 3D geometry in the real world
- Volume and surface area of various 3D shapes (cubes, rectangular prisms, cylinders, pyramids)

Sample Problem:
(CEMC-2006-Gauss7-12) A rectangular pool is 6 m wide, 12 m long and 4 m deep. If the pool is half full of water, what is the volume of water in the pool?

(A) 100 m^3 **(B)** 288 m^3 **(C)** 36 m^3 **(D)** 22 m^3 **(E)** 144 m^3

MC15N
Pre-MathCounts Number Theory

Chapter 1: Gauss Sums

- Definition of an arithmetic sequence
- Sum of the terms of an arithmetic sequence

Sample Problem:
(MathCounts-2010-School-Countdown-5) What is the sum of the 10 smallest positive multiples of three?

Chapter 2: Prime Numbers

- Definition of prime, composite numbers
- Checking if a number is prime

Sample Problem:
(AMC8-2014-4) The sum of two prime numbers is 85. What is the product of these two prime numbers?

(A) 85 **(B)** 91 **(C)** 115 **(D)** 133 **(E)** 166

Chapter 3: Prime Factorization

- Factor trees, examples of prime factorization using exponents

Sample Problem:
(CEMC-2008-Gauss7-20) The product of three *different* positive integers is 72. What is the smallest possible sum of these integers?

(A) 13 **(B)** 14 **(C)** 15 **(D)** 17 **(E)** 12

Chapter 4: Divisibility Rules

- Divisibility rules for 2, 3, 4, 5, 6, 8, 9, 10, and 11
- Divisibility by other numbers that are the product of two or more of the above numbers (e.g. 36, 99)

Sample Problem:
(Richard Spence) The four-digit number $AB37$ is divisible by 99. What is the product AB?

Chapter 5: Number of Divisors

- Determining the number of divisors of a given positive integer by computing its prime factorization

Sample Problem:
(Jonathan Sy) Find the number of divisors of 210.

Chapter 6: Sum of Divisors

- Determining the sum of divisors of a given positive integer by computing its prime factorization

Sample Problem:
(Jennifer Zhu) What is the sum of all odd factors of 162?

Chapter 7: Factoring Techniques

- Difference of squares

- Simon's Favorite Factoring Trick

Sample Problem:
(Jennifer Zhu) What is 993×1007?

Chapter 8: Number Bases

- Representing integers in bases other than 10 (emphasis on bases 2, 8, and 16)
- Converting between base 10 and a different base
- Addition and subtraction in different bases

Sample Problem:
(Richard Spence) Compute $724_8 + 145_8$. Express your answer in base 8.

Chapter 9: GCD

- Computing the greatest common factor (GCD) by factoring
- Computing the GCD using the Euclidean algorithm

Sample Problem:
(Richard Spence) Using the Euclidean algorithm, compute the greatest common divisor of 2021 and 2881.

Chapter 10: LCM

- Computing the least common multiple (LCM) by factoring
- Computing the LCM using the fact that $lcm(a, b) = ab/gcd(a, b)$

Sample Problem:
(PiMC-2018-Final Round-Team-8) What is the least common multiple of 24, 28, 32, and 36?

Chapter 11: Remainder

- Basic introduction to modular arithmetic in terms of remainders
- Simple examples using the modulo operator
- Computing remainders by finding patterns

Sample Problem:
(AMC8-2012-12) What is the units digit of 13^{2012}?

(A) 1 **(B)** 3 **(C)** 5 **(D)** 7 **(E)** 9

Chapter 12: Linear Diophantine Equations

- Solving Diophantine equations of the form $ax + by = c$, where a, b, and c are constants
- Chicken McNugget theorem

Sample Problem:
(Jennifer Zhu) Mikayla wants to spend exactly all \$20 on chips and drinks. One bag of chips costs \$3 and one drink costs \$2. How many different ways can she buy chips and/or drinks, while spending exactly \$20?

MC20A
AMC 8/MathCounts Basic Algebra

Chapter 1: Integers & Arithmetic

- Order of operations with Integers (PEMDAS)
- Introducing Variables
- Word problems using arithmetic with integers

Sample Problem:
(AMC10-2002-A6) Cindy was asked by her teacher to subtract 3 from a certain number and then divide the result by 9. Instead, she subtracted 9 and then divided the result by 3, giving an answer of 43. What would her answer have been had she worked the problem correctly?

(A) 15 **(B)** 34 **(C)** 43 **(D)** 51 **(E)** 138

Chapter 2: Fractions & Decimals

- Different types of fractions (proper/improper fractions, mixed numbers, simplest form)
- Decimals with repeating/terminating digits
- Converting between fractions and decimals
- Adding, subtracting, multiplying, dividing fractions/decimals

- Telescoping sums and products
- Word problems with fractions and decimals

Sample Problem:
(UNB-2016-Gr 9-12) A collection of coins was shared. Mary received 1/3 of the coins, Amir received 1/5 of the coins, and Samita received 1/6 of the coins. The remaining 36 coins were given to Troy. How many coins were in the entire collection?

(A) 84 **(B)** 90 **(C)** 108 **(D)** 120 **(E)** 144

Chapter 3: Percent

- Conversions between percent and fractions/decimals
- Word problems involving percent (tax, tip, interest, etc.)
- Compound Interest
- Word problems with percent

Sample Problem:
(CEMC-2000-Gauss7-13) Karl had his salary reduced by 10%. He was later promoted and his salary was increased by 10%. If his original salary was $20,000, what is his present salary?

(A) $16,200 **(B)** $19,800 **(C)** $20,000 **(D)** $20,500 **(E)** $24,000

Chapter 4: Exponents

- Basic properties of exponents (multiplying, dividing, raising an exponent to another exponent)
- Negative exponents
- Word problems with exponents

Sample Problem:
(AMC10-2002-A3) According to the standard convention for exponentiation,

$$2^{2^{2^2}} = 2^{\left(2^{\left(2^2\right)}\right)} = 2^{16} = 65,536$$

If the order in which the exponentiations are performed is changed, how many <u>other</u> values are possible?

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

Chapter 5: Radicals

- Square roots, cube roots, simplest radical form
- Negative/fractional exponents
- Rationalizing the denominator, simplifying radicals

Sample Problem:
(Richard Spence) Compute the value of $(\sqrt{12} + \sqrt{75} + \sqrt{108})^2$.

Chapter 6: Word Problems

- Converting a word problem into mathematical equations
- Solving two-unknown linear equations

Sample Problem:
(UNB-2010-Gr 9-21) Farmer Fred said to Farmer John: "If you sell me 45 hectares of land, I will have twice as much land as you." Then Farmer John said to Farmer Fred: "If you sell me 45 hectares of land, I will have just as much land as you." How many hectares of land does farmer Fred have?

(A) 135 **(B)** 180 **(C)** 225 **(D)** 270 **(E)** 315

Chapter 7: Time, Travel, Work

- Unit conversions
- Distance = Rate × Time
- Average speed, relative speed
- Problems involving the amount of work/output done

Sample Problem:
(Richard Spence) Richard goes on a 6-mile jog one morning. He jogs the first two miles at an average speed of 6 mph. He progressively slows down; his average speed during the next two miles is 4 mph. He walks the remaining two miles at an average speed of 3 mph. What is Richard's average speed, in miles per hour?

Chapter 8: Sequences-1

- Mean, median, mode, range
- Weighted average

Sample Problem:
(CEMC-2016-Gauss8-16) The mean (average) of a set of six numbers is 10. If the number 25 is removed from the set, the mean of the remaining numbers is

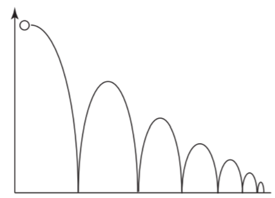
(A) 6 **(B)** 7 **(C)** 8 **(D)** 9 **(E)** 10

Chapter 9: Sequences-2

- Arithmetic and geometric sequences
- Geometric series (finite and infinite)
- Recursively defined sequences (e.g. the Fibonacci sequence)

Sample Problem:
(AMC8-2008-12) A ball is dropped from a height of 3 meters. On its first bounce it rises to a height of 2 meters. It keeps falling and bouncing to $\frac{2}{3}$ of the height it reached in the previous bounce. On which bounce will it not rise to a height of 0.5

meters?



(A) 3 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 7

Chapter 10: Functions & Operations

- Definitions of function, domain, range
- Linear functions ($f(x) = ax + b$)
- Piecewise-defined functions
- Absolute value, floor/ceiling value
- Operators

Sample Problem:
(Sean Shi) Let $a@b = (a-1)(b-1)+1$. Find $((3@4)@5)@6$.

Chapter 11: Polynomials-1

- Polynomials of a single variable
- Definitions of degree, coefficient, root
- Quadratic polynomials and the quadratic formula

Sample Problem:
(Ali Gurel) Let m and n be roots of the polynomial $x^2-28x+192$. Find a polynomial with roots $-m$ and $-n$.

Chapter 12: Polynomials-2

- Sum and product of the roots of a quadratic
- Vieta's formulas for cubic and higher degree polynomials

Sample Problem:
(Ali Gurel) Let a and g be roots of the polynomial $x^2 - 60x + 899 = 0$. What is $a^2 + g^2$?

MC20C
AMC 8/MathCounts Basic Counting

Chapter 1: Addition/Multiplication Principles

- Addition (rule of sum)
- Multiplication (rule of product)

Sample Problem:
(UNB-2008-Gr 9-16) How many ways can the numbers 1, 2, 3, 4 and 5 be placed in a line so that neither 1 nor 5 occupy either the first or the last place in the sequence?

(A) 6 **(B)** 24 **(C)** 36 **(D)** 54 **(E)** 72

Chapter 2: Permutations

- Factorials, permutations
- Counting the number of permutations of n objects taken k at a time

Sample Problem:
(CEMC-2006-Gauss8-19) Bethany, Chun, Dominic, and Emily go to the movies. They choose a row with four consecutive empty seats. If Dominic and Emily must sit beside each other, in how many different ways can the four friends sit?

(A) 6 **(B)** 5 **(C)** 12 **(D)** 30 **(E)** 3

Chapter 3: Combinations

- Difference between permutations and combinations
- How to compute combinations ("n choose k")

Sample Problem:
(Richard Spence) A math club is trying to select a subset of its members to form a committee. It notices that the number of possible 4-member committees equals the number of possible 6-member committees. How many members are in the math club?

Chapter 4: Casework

- Using casework to solve a variety of counting problems that can't be computed directly
- Use casework to break difficult problems into easier pieces

Sample Problem:
(Jafar Jafarov) A committee of five people is selected from seven men and six women. How many ways are there to select the committee so that there are at least two men and two women on the committee?

Chapter 5: Complementary Counting & Overcounting

- Applying the techniques of complementary counting or overcounting to solve problems that would be difficult otherwise

Sample Problem:
(AMC8-2016-17) An ATM password at Fred's Bank is composed of four digits from 0 to 9, with repeated digits allowable. If no password may begin with the sequence 9, 1, 1, then how many passwords are possible?

(A) 30 **(B)** 7290 **(C)** 9000 **(D)** 9990 **(E)** 9999

Chapter 6: Counting Sets

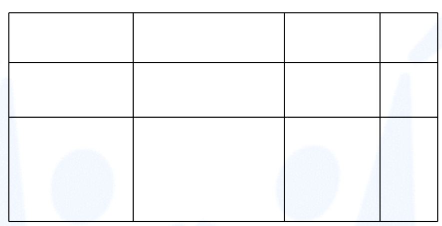
- Definitions of set, subset, size, union, and intersection
- Principle of Inclusion-Exclusion

Sample Problem:
(Jafar Jafarov) There are 25 students in a class. 12 of them are on a football team and 14 are on a soccer team. If 3 students are on neither of these teams, how many students are on both football and soccer teams?

Chapter 7: Counting Shapes & Paths

- Counting the number of paths in a lattice grid using combinations and permutations
- Counting shapes or paths systematically (e.g. without counting manually)

Sample Problem:
(Richard Spence) How many rectangles of any size are in the figure below?



Chapter 8: Counting with Digits

- Various counting problems involving digits
- Palindromic numbers

Sample Problem:
(BmMT-2016-Team-8) A seven digit number is called "bad" if exactly four of its digits are 0 and the rest are odd. How many seven digit numbers are bad?

Chapter 9: Stars and Bars

- Applying the stars and bars (or "balls and boxes") technique to solve various counting problems

Sample Problem:
(Victor Hakim) How many positive integer solutions (x, y, z, w) are there to $x + y + z + w = 15$?

Chapter 10: Binomial & Pascal's Triangle

- Binomial theorem (expanding $(x + y)^n$)
- Pascal's triangle

Sample Problem:
(Richard Spence) Simplify

$$\binom{100}{0} + 2\binom{100}{1} + 4\binom{100}{2} + 8\binom{100}{3} + \ldots + 2^{100}\binom{100}{100}.$$

Chapter 11: Probability-1

- Definition, of probability
- Sample space, independent/dependent events, disjoint events

Sample Problem:
(Richard Spence) Four fair six-sided dice are rolled. What is the probability that the largest number rolled is at least 4? Express your answer as a common fraction in reduced form.

Chapter 12: Probability-2

- Expected value and linearity of expectation
- Conditional probability, Bayes' theorem
- Geometric probability

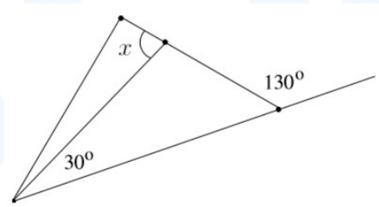
Sample Problem:
(Victor Hakim) x and y are two positive real numbers chosen randomly and uniformly in the interval $[0, 2]$. What is the probability that $x^2 + y^2 \geq 1$ and $y \geq x$?

MC20G
AMC 8/MathCounts Basic Geometry

Chapter 1: Angles

- Definitions of acute, right, obtuse, complementary, and supplementary angles
- Parallel, perpendicular, and transversal lines
- Sum of the degree measures in a triangle, different types of triangles
- Inscribed angles and arcs in circles

Sample Problem: (UNB-2018-Gr 9-3) Find the measure of the angle labeled x in the diagram.



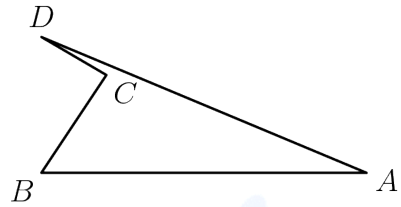
(A) 70° **(B)** 75° **(C)** 80° **(D)** 100° **(E)** 160°

Chapter 2: Special Triangles

- 30-60-90 and 45-45-90 triangles

- Pythagorean theorem and Pythagorean triples

Sample Problem: (AMC8-2017-18) In the non-convex quadrilateral $ABCD$ shown below, $\angle BCD$ is a right angle, $AB = 12$, $BC = 4$, $CD = 3$, and $AD = 13$.



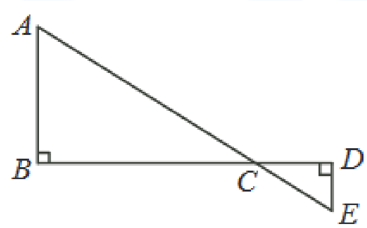
What is the area of quadrilateral $ABCD$?

(A) 12 **(B)** 24 **(C)** 26 **(D)** 30 **(E)** 36

Chapter 3: Similarity

- Congruence and similarity axioms (SSS, SAS, ASA, AA)
- SSA is not a congruence axiom
- Angle bisector theorem

Sample Problem: (CEMC-2011-Gauss8-15) In the diagram, AE and BD are straight lines that intersect at C. If $BD = 16$, $AB = 9$, $CE = 5$, and $DE = 3$, then the length of AC is



(A) 11 **(B)** 12 **(C)** 15 **(D)** 17 **(E)** 16

Chapter 4: Length-1

- Perimeter of polygons

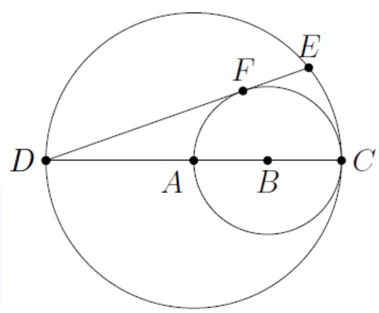
- Triangle inequality
- Review of the Pythagorean theorem

Sample Problem:
(A-Star) A right triangle has legs 8 cm and 15 cm. Find the shortest altitude of this triangle, in centimeters. Express your answer as a common fraction in reduced form.

Chapter 5: Length-2

- Circumference of a circle
- Power of a point
- Inscribed and circumscribed circles of a triangle
- Ravi substitution

Sample Problem:
(PPP Vol8 p36 q10) In the diagram, DC is a diameter of the larger circle centered at A, and AC is a diameter of the smaller circle centered at B. If DE is tangent to the smaller circle at F, and $DC = 12$, determine the length of DE.



Chapter 6: Length-3

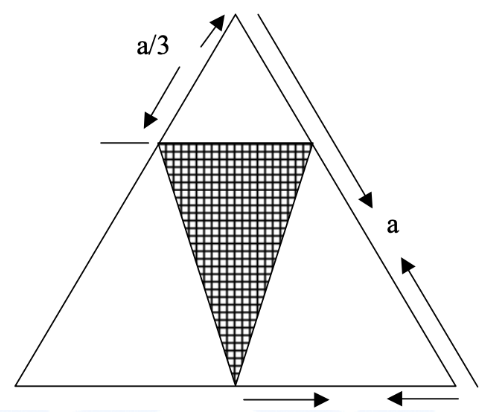
- Introduction to the mass points technique using physics concepts (levers, torque)
- Ceva's theorem and Menelaus' theorem

Sample Problem:
(Ali Gurel) P is a point inside triangle $\triangle ABC$. The lines AP, BP, and CP intersect the sides of the triangle at D, E, and F, respectively. If $AP = PD$ and $BD = 2DC$, what is BF/FA?

Chapter 7: Area-1

- Unit conversions (e.g. square feet to square yards)
- Areas of simple polygons (squares, rectangles, triangles, trapezoids)
- Other formulas for the area of a triangle, including Heron's

Sample Problem:
(UNB-2000-Gr 9-22) In the figure below, the area of the shaded triangle is $2\sqrt{3}$. If the large triangle and the small upper triangle are equilateral, what is the value of a?



(A) 2 **(B)** 2.5 **(C)** 3 **(D)** 6 **(E)** None of these

Chapter 8: Area-2

- Area of a circle and sector

Sample Problem:
(Classic) Three circles with radius 1 meter are pairwise tangent to each other. Find the area that is enclosed in between the three circles.

Chapter 9: Analytic Geometry-1

- Cartesian coordinate system (2 dimensions)
- Slope-intercept and point-slope form of a line
- Midpoint and distance formula
- Solving geometry problems by using coordinates

Sample Problem:
(BmMT-2012-Ciphering-26) The lines $y = 3x$ and $x = 4$ form a right triangle with the x-axis. Find the slope of a line through the origin that bisects the triangle into two portions of equal area.

Chapter 10: Analytic Geometry-2

- Reflecting/rotating a point in the coordinate plane
- General equation of a circle in the coordinate plane
- Area of a polygon with Shoelace formula

Sample Problem:
(PPP Vol5 p26 q11) Find the coordinates of all points in the Cartesian plane that are equidistant from the x-axis, y-axis, and the point $(2, 1)$.

Chapter 11: 3D-1

- Applications of 3D geometry in the real world
- Applying 2D geometry techniques to 3D space, 3D distance formula
- Surface area of various polyhedra, cylinders, cones, spheres

Sample Problem:
(BmMT-2012-Ciphering-20) What is the surface area of a cube inscribed in a sphere with surface area 8π?

Chapter 12: 3D-2

- Volume of various 3D shapes (polyhedra, cylinders, cones, spheres)
- Volume of more complex shapes

Sample Problem:
(BmMT-2012-Ciphering-24) An 8×11 sheet of paper is rolled up so that the 11-inch edges align. Find the volume of the resulting cylinder.

MC20N
AMC 8/MathCounts Basic Number Theory

Chapter 1: Gauss Sums

- Sums of arithmetic sequences (e.g. sum of the first n positive integers)
- Sum of the first n perfect squares, cubes

Sample Problem:
(UNB-2002-Gr 9-22) The value of the expression $1-2-3+4+5-6-7+8+9-\ldots+76+77-78-79$ is equal to

(A) -98 **(B)** -80 **(C)** -60 **(D)** 40 **(E)** 80

Chapter 2: Primes & Prime Factorization

- Definition of divisibility
- Fundamental Theorem of Arithmetic
- Determining if a number is prime or not
- Legendre's formula

Sample Problem:
(Sean Shi) The ages of Monica's three children are between 12 and 17. The product of their ages is 3120. What is the sum of their ages?

Chapter 3: Divisibility Rules

- Divisibility rules for all positive integers up to and including 11

Sample Problem:
(Jane Ahn) What is the smallest number greater than 200 that is divisible by both 14 and 21?

Chapter 4: Number of Divisors

- Determining the number of divisors of a positive integer n using the prime factorization of n
- Multiplicative functions

Sample Problem:
(Jennifer Zhu) What is the difference between the numbers of factors in 288 and in 144?

Chapter 5: Sum of Divisors

- Definition of $\sigma(n)$
- Determining the sum of divisors of a number n using the prime factorization of n

Sample Problem:
(Richard Spence) What is the sum of the divisors of 496?

Chapter 6: Factoring Techniques

- Difference of squares
- Simon's Favorite Factoring Trick (SFFT)
- Sum of cubes, difference of cubes
- Sophie-Germain identity

Sample Problem:
(Jonathan Sy) How many distinct prime factors does $17^4 - 4^4$ have?

Chapter 7: Number Bases

- Representing numbers in different bases
- Converting numbers between bases (emphasis on base 2, 8, and 16)
- Arithmetic in different bases

Sample Problem:
(Jacob Klegar) How many positive integers have the same number of digits in base 5 and base 9?

Chapter 8: GCD & LCM

- Computing the GCD and LCM of two or more numbers
- Euclidean algorithm
- Relation between gcd and lcm ($lcm(a, b) = ab/gcd(a, b)$)

Sample Problem:
(Nathan Zhang) Find the greatest common divisor of 2613 and 637.

Chapter 9: Modular Arithmetic

- Introduction to the congruence operator ($a \equiv b \pmod{m}$)
- Basic properties of modulo (reflexive, symmetric, transitive)
- Computing remainders by finding patterns
- Proof of the divisibility rules for 3, 9, and 11

Sample Problem:
(Kevin Chang) Find the last two digits of 7^{2015}.

Chapter 10: Fermat's Little Theorem

- Applying Fermat's little theorem to find the remainder when a power is divided by a prime

Sample Problem:
(BmMT-2012-Ciphering-31) What is the remainder when 19^{19} is divided by 17?

Chapter 11: Chinese Remainder Theorem

- Applying the Chinese remainder theorem to basic modular arithmetic problems
- Solving basic systems of congruences

Sample Problem:
(Victor Hakim) If a large batch of donuts is arranged into boxes of 10, eight are left over. If arranged into boxes of 12, ten are left over. If arranged into boxes of 14, twelve are left over. Given that there are fewer than 500 donuts, how many donuts are in the batch?

Chapter 12: Diophantine Equations

- Solving linear Diophantine equations of the form $ax + by = c$
- Chicken McNugget theorem
- Bézout's identity
- Pythagorean triples
- Using modular arithmetic to show that a Diophantine equation has no solutions

Sample Problem:
(BmMT-2012-Team-4) How many solutions (x, y) in the positive integers are there to $3x + 7y = 1337$?

MC25A
AMC 8/MathCounts Advanced Algebra

Chapter 1: Integers & Arithmetic

- Order of operations with Integers (PEMDAS)
- Introducing Variables
- Word problems using arithmetic with integers

Sample Problem:
(AMC10-2004-A6) Bertha has 6 daughters and no sons. Some of her daughters have 6 daughters, and the rest have none. Bertha has a total of 30 daughters and granddaughters, and no great-granddaughters. How many of Bertha's daughters and granddaughters have no daughters?

(A) 22 **(B)** 23 **(C)** 24 **(D)** 25 **(E)** 26

Chapter 2: Fractions & Decimals

- Different types of fractions (proper/improper fractions, mixed numbers, simplest form)
- Decimals with repeating/terminating digits
- Converting between fractions and decimals
- Adding, subtracting, multiplying, dividing fractions/decimals

- Telescoping sums and products
- Word problems with fractions and decimals

Sample Problem:
(AMC8-2010-21) Hui is an avid reader. She bought a copy of the best seller *Math is Beautiful*. On the first day, she read $\frac{1}{5}$ of the pages plus 12 more, and on the second day she read $\frac{1}{4}$ of the remaining pages plus 15 more. On the third day she read $\frac{1}{3}$ of the remaining pages plus 18 more. She then realizes she has 62 pages left, which she finishes the next day. How many pages are in this book?

(A) 120 **(B)** 180 **(C)** 240 **(D)** 300 **(E)** 360

Chapter 3: Percent

- Conversions between percent and fractions/decimals
- Word problems involving percent (tax, tip, interest, etc.)
- Compound Interest
- Word problems with percent

Sample Problem:
(AMC10-2008-A8) Heather compares the price of a new computer at two different stores. Store A offers 15% off the sticker price followed by a $90 rebate, and store B offers 25% off the same sticker price with no rebate. Heather saves $15 by buying the computer at store A instead of store B. What is the sticker price of the computer, in dollars?

(A) 750 **(B)** 900 **(C)** 1000 **(D)** 1050 **(E)** 1500

Chapter 4: Exponents

- Basic properties of exponents (multiplying, dividing, raising an exponent to another exponent)
- Negative exponents
- Word problems with exponents

Sample Problem:
(AMC8-2010-24) What is the correct ordering of the three numbers, 10^8, 5^{12}, and 2^{24}?

(A) $2^{24} < 10^8 < 5^{12}$ **(B)** $2^{24} < 5^{12} < 10^8$ **(C)** $5^{12} < 2^{24} < 10^8$ **(D)** $10^8 < 5^{12} < 2^{24}$ **(E)** $10^8 < 2^{24} < 5^{12}$

Chapter 5: Radicals

- Square roots, cube roots, simplest radical form
- Negative/fractional exponents
- Rationalizing the denominator, simplifying radicals

Sample Problem:
(Jennifer Zhang) Simplify $\dfrac{\sqrt{63}}{\sqrt{315}+\sqrt{140}}$.

Chapter 6: Word Problems

- Converting a word problem into mathematical equations
- Solving two-unknown linear equations

Sample Problem:
(UNB-2008-Gr 9-7) Marina has a bank containing only pennies and nickels. If the pennies were nickels and the nickels were pennies, she would have exactly \$1.00 more. If the total value of the money in her bank is \$1.75, how many pennies does Marina have?

(A) 25 **(B)** 30 **(C)** 40 **(D)** 50 **(E)** Not enough information

Chapter 7: Time, Travel, Work

- Unit conversions
- Distance = Rate × Time

- Average speed, relative speed
- Problems involving the amount of work/output done

Sample Problem:
(AMC10-2008-A6) A triathlete competes in a triathlon in which the swimming, biking, and running segments are all of the same length. The triathlete swims at a rate of 3 kilometers per hour, bikes at a rate of 20 kilometers per hour, and runs at a rate of 10 kilometers per hour. Which of the following is closest to the triathlete's average speed, in kilometers per hour, for the entire race?

(A) 3 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 7

Chapter 8: Sequences-1

- Mean, median, mode, range
- Weighted average

Sample Problem:
(Kevin Chang) Given a set of four positive integers with median 20 and mean 23, what is the smallest possible range of the set?

Chapter 9: Sequences-2

- Arithmetic and geometric sequences
- Geometric series (finite and infinite)
- Recursively defined sequences (e.g. the Fibonacci sequence)

Sample Problem:
(Kevin Chang) If the angles of an octagon are all integers and form an arithmetic sequence, find the sum of all possible values of the smallest angle. Express your answer in degrees.

Chapter 10: Functions & Operations

- Definitions of function, domain, range
- Linear functions ($f(x) = ax + b$)
- Piecewise-defined functions
- Absolute value, floor/ceiling value
- Operators

Sample Problem:
(Kevin Chang) A tank is to be filled with water. Adding 130 gallons to an empty tank fills 52% of the tank. How many gallons does the tank contain when it is completely full?

Chapter 11: Polynomials-1

- Polynomials of a single variable
- Definitions of degree, coefficient, root
- Quadratic polynomials and the quadratic formula

Sample Problem:
(Kevin Chang) Define $a \otimes b = ab - a - b$ for real numbers a, b. Then evaluate

$$(((100 \otimes 99) \otimes 98) \otimes \cdots) \otimes 1.$$

Chapter 12: Polynomials-2

- Sum and product of the roots of a quadratic
- Vieta's formulas for cubic and higher degree polynomials

Sample Problem:
(Ali Gurel) Let m and n be roots of the polynomial $x^2 - 60x + 864 = 0$. Find a polynomial with roots $m + 1$ and $n + 1$.

MC25C
AMC 8/MathCounts Advanced Counting

Chapter 1: Addition/Multiplication Principles

- Addition (rule of sum)
- Multiplication (rule of product)

Sample Problem:
(Richard Spence) How many positive divisors of 10! are multiples of 10?

Chapter 2: Permutations

- Factorials, permutations
- Counting the number of permutations of n objects taken k at a time

Sample Problem:
(Richard Spence) The sixteen students at $C^\star$ Math Camp class split up into four groups: the Teleporters, the Transformers, the Timebenders, and the Mindbenders. Each group then decides one person to solve the number theory problems, one person to solve the algebra problems, one person to solve the geometry problems, and one person to solve the combinatorics problems, such that each student in each group solves exactly one type of problem. In how many ways can this be done?

Chapter 3: Combinations

- Difference between permutations and combinations
- How to compute combinations ("n choose k")

Sample Problem:
(A-Star) Two lines intersect at a point. 3 points are given on the first line and 4 points are given on the second line. None of the seven points are the intersection point of two lines. Find the number of triangles whose vertices are among these 7 given points.

Chapter 4: Casework

- Using casework to solve a variety of counting problems that can't be computed directly
- Use casework to break difficult problems into easier pieces

Sample Problem:
(CEMC-2012-Gauss8-24) Stones are numbered 1, 2, 3, 4, 5, 6, 7, 8, 9, 10. Three groups of stones can be selected so that the sum of each group is 11. For example, one arrangement is {1, 10}, {2, 3, 6}, {4, 7}. Including the example, how many arrangements are possible?

(A) 13 **(B)** 16 **(C)** 11 **(D)** 12 **(E)** 15

Chapter 5: Complementary Counting & Overcounting

- Applying the techniques of complementary counting or overcounting to solve problems that would be difficult otherwise

Sample Problem:
(Richard Spence) There are ten students in a class. How many ways can the teacher pair up the students into five pairs of two students each? The order of the students in each pair and the ordering of the five pairs does not matter.

Chapter 6: Counting Sets

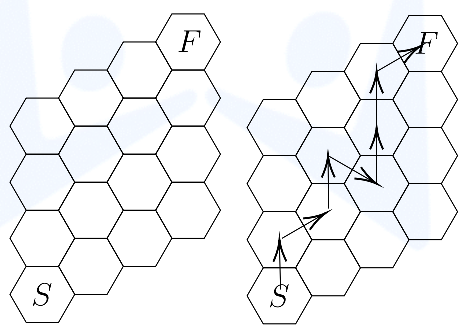
- Definitions of set, subset, size, union, and intersection
- Principle of Inclusion-Exclusion

Sample Problem:
(Sean Shi) There are 150 people at a camp. 70 of them play chess, 50 play cards, 50 play checkers, 20 play both chess and checkers, 14 play both chess and cards, and 16 play both checkers and cards. In addition, 4 play all three games. How many people play none of the games?

Chapter 7: Counting Shapes & Paths

- Counting the number of paths in a lattice grid using combinations and permutations
- Counting shapes or paths systematically (e.g. without counting manually)

Sample Problem:
(Richard Spence) A frog starts at S on the hexagonal grid of 16 tiles shown below. On any move, it can hop from its current hexagon to the hexagon above it, or either hexagon(s) on its right. An example sequence of moves is shown to the right. In how many different ways can the frog reach F using any such sequence of moves?



Chapter 8: Counting with Digits

- Various counting problems involving digits
- Palindromic numbers

Sample Problem:
(Kevin Chang) How many 3-digit numbers are there such that the hundreds digit is equal to the average of the tens and units digits?

Chapter 9: Stars and Bars

- Applying the stars and bars (or "balls and boxes") technique to solve various counting problems

Sample Problem:
(Sean Shi) How many integer solutions (w, x, y, z) are there to $w + x + y + z = 12$ such that $0 \leq w, x, y, z$; $x \leq 1$; and y is even?

Chapter 10: Binomial & Pascal's Triangle

- Binomial theorem (expanding $(x + y)^n$)
- Pascal's triangle

Sample Problem:
(Classic) Give a closed formula for the sum

$$1\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \cdots + n\binom{n}{n}.$$

Chapter 11: Probability-1

- Definition, of probability
- Sample space, independent/dependent events, disjoint events

Sample Problem:
(BmMT-2016-Individual-16) Alice rolls one pair of 6-sided dice, and Bob rolls another pair of 6-sided dice. What is the probability that at least one of Alice's dice shows the same number as at least one of Bob's dice?

Chapter 12: Probability-2

- Expected value and linearity of expectation
- Conditional probability, Bayes' theorem
- Geometric probability

Sample Problem:
(Richard Spence) In a certain country, exactly 1% of the population has a disease called algebritis. A certain drug test for algebritis claims a 99% accuracy; i.e. it returns a correct positive (or negative) result with probability 0.99.
A random participant from this country is selected. Given that his algebritis test was positive, what is the probability that he has algebritis?

MC25G
AMC 8/MathCounts Advanced Geometry

Chapter 1: Angles

- Definitions of acute, right, obtuse, complementary, and supplementary angles
- Parallel, perpendicular, and transversal lines
- Sum of the degree measures in a triangle, different types of triangles
- Inscribed angles and arcs in circles

Sample Problem:
(Sean Shi) In triangle ABC, the measure of angle A is 42. The angle bisectors of angles B and C meet at I. Find the measure of angle BIC.

Chapter 2: Special Triangles

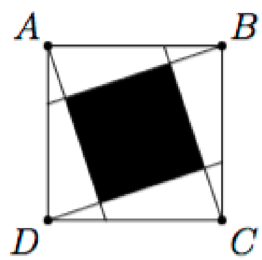
- 30-60-90 and 45-45-90 triangles
- Pythagorean theorem and Pythagorean triples

Sample Problem:
(Wanlin Li) Quadrilateral $ABCD$ has right angles at B and D. If $AC = 2\sqrt{6}$, $AB = \sqrt{6}$, and $AD = 2\sqrt{3}$, find the area of $ABCD$ in simplest radical form.

Chapter 3: Similarity

- Congruence and similarity axioms (SSS, SAS, ASA, AA)
- SSA is not a congruence axiom
- Angle bisector theorem

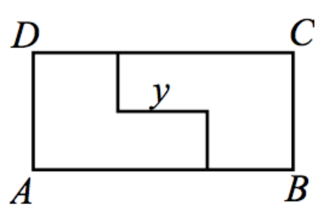
Sample Problem:
(BmMT-2012-Team-8) As pictured, lines are drawn from the vertices of a unit square to an opposite trisection point. If each triangle has legs with ratio 3:1, what is the area of the shaded region?



Chapter 4: Length-1

- Perimeter of polygons
- Triangle inequality
- Review of the Pythagorean theorem

Sample Problem:
(AMC10-2006-A7) The 8×18 rectangle $ABCD$ is cut into two congruent hexagons, as shown, in such a way that the two hexagons can be repositioned without overlap to form a square. What is y?

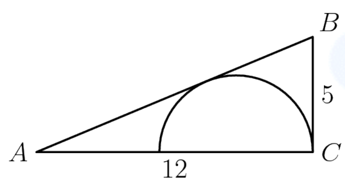


(A) 6 **(B)** 7 **(C)** 8 **(D)** 9 **(E)** 10

Chapter 5: Length-2

- Circumference of a circle
- Power of a point
- Inscribed and circumscribed circles of a triangle
- Ravi substitution

Sample Problem:
(AMC8-2017-22) In the right triangle ABC, $AC = 12$, $BC = 5$, and angle C is a right angle. A semicircle is inscribed in the triangle as shown. What is the radius of the semicircle?



(A) $\frac{7}{6}$ **(B)** $\frac{13}{5}$ **(C)** $\frac{59}{18}$ **(D)** $\frac{10}{3}$ **(E)** $\frac{60}{13}$

Chapter 6: Length-3

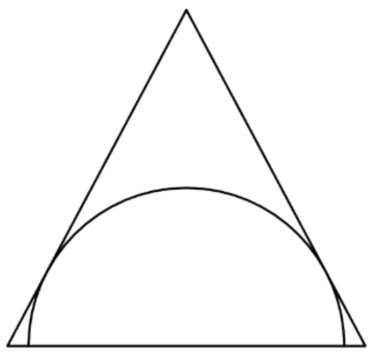
- Introduction to the mass points technique using physics concepts (levers, torque)
- Ceva's theorem and Menelaus' theorem

Sample Problem:
(Ali Gurel) In $\triangle ABC$, points D, E, and F are on BC, CA, and AB, respectively. Suppose the segments AD, BE, and CF intersect at P.
If $AF/AB = 1/3$ and $AE/AC = 1/4$, what is AP/AD?

Chapter 7: Area-1

- Unit conversions (e.g. square feet to square yards)
- Areas of simple polygons (squares, rectangles, triangles, trapezoids)
- Other formulas for the area of a triangle, including Heron's

Sample Problem:
(AMC8-2016-25) A semicircle is inscribed in an isosceles triangle with base 16 and height 15 so that the diameter of the semicircle is contained in the base of the triangle as shown. What is the radius of the semicircle?

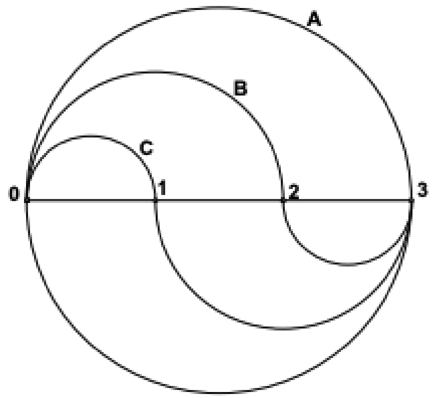


(A) $4\sqrt{3}$ **(B)** $\frac{120}{17}$ **(C)** 10 **(D)** $\frac{17\sqrt{2}}{2}$ **(E)** $\frac{17\sqrt{3}}{2}$

Chapter 8: Area-2

- Area of a circle and sector

Sample Problem:
(UNB-2008-Gr 9-24) A is a circle whose diameter is equal to 3 units. Curves B and C are respectively made from one half-circle of diameter equal to 1 unit and one half-circle of diameter equal to 2 units. What is the area of the region located between curves B and C?



(A) $\frac{3}{4}$ **(B)** $\frac{3\pi}{4}$ **(C)** 3 **(D)** 3π **(E)** None of these

Chapter 9: Analytic Geometry-1

- Cartesian coordinate system (2 dimensions)
- Slope-intercept and point-slope form of a line
- Midpoint and distance formula
- Solving geometry problems by using coordinates

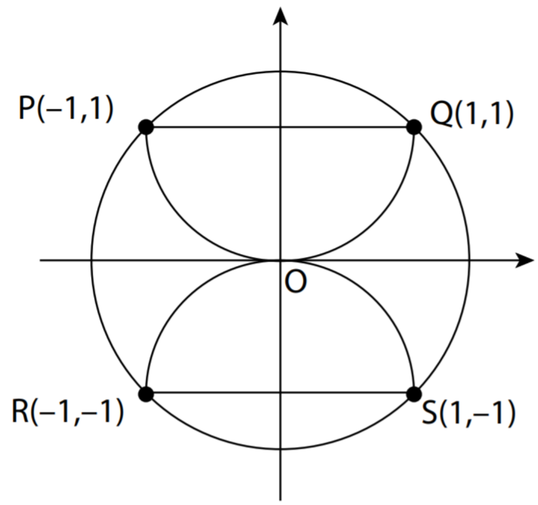
Sample Problem:
(AMC12-2018-B3) A line with slope 2 intersects a line with slope 6 at the point $(40, 30)$. What is the distance between the x-intercepts of these two lines?

(A) 5 **(B)** 10 **(C)** 20 **(D)** 25 **(E)** 50

Chapter 10: Analytic Geometry-2

- Reflecting/rotating a point in the coordinate plane
- General equation of a circle in the coordinate plane
- Area of a polygon with Shoelace formula

Sample Problem:
(AMC8-2010-23) Semicircles POQ and ROS pass through the center of circle O. What is the ratio of the combined areas of the two semicircles to the area of circle O ?



(A) $\frac{\sqrt{2}}{4}$ **(B)** $\frac{1}{2}$ **(C)** $\frac{2}{\pi}$ **(D)** $\frac{2}{3}$ **(E)** $\frac{\sqrt{2}}{2}$

Chapter 11: 3D-1

- Applications of 3D geometry in the real world
- Applying 2D geometry techniques to 3D space, 3D distance formula
- Surface area of various polyhedra, cylinders, cones, spheres

Sample Problem:
(Alexander Parr) A paper towel roll company is designing a cardboard paper towel roll. It has a circumference of 5 and a height of 48. There is a crease that starts at the bottom and spirals around the tube 4 times before reaching the top. What is the length of the spiral?

Chapter 12: 3D-2

- Volume of various 3D shapes (polyhedra, cylinders, cones, spheres)

- Volume of more complex shapes

Sample Problem:
(Alexander Parr) A cube is inscribed inside of a sphere with a radius of 1 cm. Find the volume of the region inside the sphere but outside of the cube.

MC25N
AMC 8/MathCounts Advanced Number Theory

Chapter 1: Gauss Sums

- Sums of arithmetic sequences (e.g. sum of the first n positive integers)
- Sum of the first n perfect squares, cubes

Sample Problem:
(Richard Spence) The sum $1^3 + 2^3 + 3^3 + \ldots + n^3$ is equal to a perfect fourth power, where $n \geq 1$. What is the smallest possible value of n?

Chapter 2: Primes & Prime Factorization

- Definition of divisibility
- Fundamental Theorem of Arithmetic
- Determining if a number is prime or not
- Legendre's formula

Sample Problem:
(AMC10-2002-B6) For how many positive integers n is $n^2 - 3n + 2$ a prime number?

(A) none **(B)** one **(C)** two **(D)** more than two, but finitely many **(E)** infinitely many

Chapter 3: Divisibility Rules

- Divisibility rules for all positive integers up to and including 11

Sample Problem:
(Sean Shi) What is the smallest five-digit positive integer divisible by 5 with digit sum 27?

Chapter 4: Number of Divisors

- Determining the number of divisors of a positive integer n using the prime factorization of n
- Multiplicative functions

Sample Problem:
(Tiancheng Qin) Given that b and n are both positive integers at most 15, what is the greatest number of divisors that b^n can have?

Chapter 5: Sum of Divisors

- Definition of $\sigma(n)$
- Determining the sum of divisors of a number n using the prime factorization of n

Sample Problem:
(Ali Gurel) Find the smallest two consecutive squares whose sum of divisors are the same.

Chapter 6: Factoring Techniques

- Difference of squares
- Simon's Favorite Factoring Trick (SFFT)
- Sum of cubes, difference of cubes
- Sophie-Germain identity

Sample Problem:
(Ali Gurel) Find the sum of prime divisors of 4891.

Chapter 7: Number Bases

- Representing numbers in different bases
- Converting numbers between bases (emphasis on base 2, 8, and 16)
- Arithmetic in different bases

Sample Problem:
(Nathan Zhang) Find the base-10 value of $11_2 + 22_3 + 33_4 + \ldots + 99_{10}$.

Chapter 8: GCD & LCM

- Computing the GCD and LCM of two or more numbers
- Euclidean algorithm
- Relation between gcd and lcm ($lcm(a, b) = ab/gcd(a, b)$)

Sample Problem:
(Ali Gurel) How many pairs of ordered positive integers (a, b) are there such that $\text{lcm}(a, b) = 48$ and $\text{gcd}(a, b) = 4$?

Chapter 9: Modular Arithmetic

- Introduction to the congruence operator ($a \equiv b \pmod{m}$)
- Basic properties of modulo (reflexive, symmetric, transitive)
- Computing remainders by finding patterns
- Proof of the divisibility rules for 3, 9, and 11

Sample Problem:
(Kevin Chang) The Fibonacci sequence is the sequence $1, 1, 2, 3, 5, \ldots,$ where each term after the second is the sum of the previous two terms. What is the units digit of the 10^6th Fibonacci number?

Chapter 10: Fermat's Little Theorem

- Applying Fermat's little theorem to find the remainder when a power is divided by a prime

Sample Problem:
(Richard Spence) What is the units digit of $1^{12} + 2^{12} + 3^{12} + \ldots + 2019^{12}$?

Chapter 11: Chinese Remainder Theorem

- Applying the Chinese remainder theorem to basic modular arithmetic problems
- Solving basic systems of congruences

Sample Problem:
(Kevin Chang) How many integers between 1 and 2520, inclusive, are divisible by 36, but not by 5 or 7?

Chapter 12: Diophantine Equations

- Solving linear Diophantine equations of the form $ax + by = c$
- Chicken McNugget theorem
- Bézout's identity
- Pythagorean triples
- Using modular arithmetic to show that a Diophantine equation has no solutions

Sample Problem:
(Richard Spence) Hexagonal-shaped tubing is sold in packages of 7 and 19 tubes. What is the smallest number k such that for any $n \geq k$, I can always buy exactly n tubes?

MC30A
AMC 10/12 Basic Algebra

Chapter 1: Arithmetic

- Word problems using arithmetic with integers, fractions, decimals, and percent
- Decimals with repeating/terminating digits
- Rational/Irrational numbers

Sample Problem: (AMC10-2009-A5) What is the sum of the digits of the square of $111,111,111$?

(A) 18 **(B)** 27 **(C)** 45 **(D)** 63 **(E)** 81

Chapter 2: Exponents & Radicals

- Properties of exponents and radicals
- Negative/fractional exponents
- Rationalizing the denominator, simplifying radicals
- Using conjugates of radicals

Sample Problem:
(Lehigh MC-2008-10) Simplify $\sqrt{19+\sqrt{297}}-\sqrt{19-\sqrt{297}}$.

Chapter 3: Word Problems & System of Equations

- Word problems, systems of equations in two or more variables

Sample Problem:
(SMT-2012-General-4) Steve works 40 hours a week at his new job. He usually gets paid 8 dollars an hour, but if he works for more than 8 hours on a given day, he earns 12 dollars an hour for every additional hour over 8 hours. If x is the maximum number of dollars that Steve can earn in one week by working exactly 40 hours, and y is the minimum number of dollars that Steve can earn in one week by working exactly 40 hours, what is $x - y$?

Chapter 4: Time, Travel, Work

- Distance = Rate × Time, average speed
- Harmonic mean
- Relative speed
- Rate/Work Problems

Sample Problem:
(AMC10-2002-A12) Mr. Earl E. Bird leaves his house for work at exactly 8:00 A.M. every morning. When he averages 40 miles per hour, he arrives at his workplace three minutes late. When he averages 60 miles per hour, he arrives three minutes early. At what average speed, in miles per hour, should Mr. Bird drive to arrive at his workplace precisely on time?

(A) 45 **(B)** 48 **(C)** 50 **(D)** 55 **(E)** 58

Chapter 5: Sequences-1

- Mean, median, mode, range
- Arithmetic and geometric sequences
- Geometric series formula and derivation

Sample Problem:
(AMC10-2010-B17) Every high school in the city of Euclid sent a team of 3 students to a math contest. Each participant in the contest received a different score. Andrea's score was the median among all students, and hers was the highest score on her team. Andrea's teammates Beth and Carla placed 37^{th} and 64^{th}, respectively. How many schools are in the city?

(A) 22 **(B)** 23 **(C)** 24 **(D)** 25 **(E)** 26

Chapter 6: Sequences-2

- Recurrent sequences
- Finding the general term via patterns

Sample Problem:
(Math Day at the Beach-2018-Individual-14) Form the sequence such that $x_1 = x_2 = 1$, and for $n > 2$, $x_n = x_{n-1}^2 + x_{n-2}$. Of the numbers $x_1,\ x_2, \ldots, x_{2018}$, how many are divisible by 3?

Chapter 7: Functions & Operations

- Definitions of function, domain, codomain/range
- Injective, surjective, bijective functions
- Inverse functions
- Operators
- Simple functional equations

Sample Problem:
(Math Day at the Beach-2014-Individual-18) Compute $\sum_{n=1}^{99} \lfloor 0.67n \rfloor$,where the notation $\lfloor x \rfloor$ means the greatest integer that is less than or equal to x.

Chapter 8: Polynomials-1

- Polynomials of a single variable; definitions of degree, root, etc.
- Solving for the roots of a quadratic by factoring, completing the square, or quadratic formula
- Rational root theorem
- Fundamental theorem of algebra
- Less emphasis on complex numbers (Chapter 12)

Sample Problem:
(HMMT Nov-2012-Guts-15) Find the area of the region in the xy-plane consisting of all points (a, b) such that the quadratic $ax^2 + 2(a + b - 7)x + 2b = 0$ has fewer than two real solutions for x.

Chapter 9: Polynomials-2

- Generalized Vieta's formulas
- Manipulation of symmetric sums to produce other expressions

Sample Problem:
(Justin Stevens) Find $(3 - r)(3 - s)(3 - t)$ if r, s, and t are the roots of $f(x) = 3x^3 - 9x^2 + 3x - 7$. Express your answer as a common fraction in reduced form.

Chapter 10: Trigonometry

- Review of trigonometric functions (sin, cos, tan, csc, sec, cot)
- More emphasis on trigonometric identities (addition and multiple-angle formulae)
- Solving algebra problems via trig substitution

Sample Problem:
(Richard Spence) How many solutions $\theta \in [0, 2\pi)$ are there such that $\sin\theta = \sin 6\theta$?

Chapter 11: Logarithm

- Definition of a logarithm in base b, simple logarithmic identities (change-of-base formula, addition/subtraction of logarithms)
- Natural logarithms, the number e
- Applications: Binary search, merge sort example

Sample Problem:
(AMC12-2018-A14) The solution to the equation $\log_{3x} 4 = \log_{2x} 8$, where x is a positive real number other than $\frac{1}{3}$ or $\frac{1}{2}$, can be written as $\frac{p}{q}$, where p and q are relatively prime positive integers. What is $p + q$?

(A) 5 **(B)** 13 **(C)** 17 **(D)** 31 **(E)** 35

Chapter 12: Complex Numbers

- More rigorous introduction to complex numbers
- Review of the Fundamental Theorem of Algebra, conjugate root theorem
- Polar form, Euler's formula, de Moivre's formula
- Roots of unity

Sample Problem:
(BMT-2016-Individual-5) Positive integers x, y, z satisfy $(x + yi)^2 - 46i = z$. What is $x + y + z$?

MC30C
AMC 10/12 Basic Counting

Chapter 1: Counting Basics

- Addition/multiplication principles
- Permutations, combinations, binomial coefficients

Sample Problem:
(Justin Stevens) How many subsets does the set of odd numbers $\{1, 3, 5, 7, 9, \cdots, 19\}$ have?

Chapter 2: Casework

- Solving a variety of counting problems using casework
- Use casework to break difficult problems into easier pieces

Sample Problem:
(AMC12-2014-A13) A fancy bed and breakfast inn has 5 rooms, each with a distinctive color-coded decor. One day 5 friends arrive to spend the night. There are no other guests that night. The friends can room in any combination they wish, but with no more than 2 friends per room. In how many ways can the innkeeper assign the guests to the rooms?

(A) 2100 **(B)** 2220 **(C)** 3000 **(D)** 3120 **(E)** 3125

Chapter 3: Complementary Counting & Overcounting

- Solving counting problems using the techniques of complementary counting and/or overcounting

Sample Problem:

Chapter 4: Counting Sets

- Definitions of set, cardinality, union, intersection
- Principle of inclusion-exclusion for two or more sets

Sample Problem:
(Justin Stevens) There are 140 students in my high school. 70 of them play basketball, 100 of them play soccer, and 30 play hockey. 44 play both soccer and basketball, 12 play basketball and hockey, and 9 play soccer and hockey. How many students play all three sports?

Chapter 5: Counting with Digits

- Solving a variety of counting problems involving digits of a number
- Counting palindromes

Sample Problem:
(Jamie Gu) How many 5-digit positive integers have exactly three 5's?

Chapter 6: Path Counting & Bijections

- Definitions of injective, surjective, and bijective functions
- Examples of bijections between two infinite sets (e.g. the set of whole numbers and the set of integers)
- Solving counting problems by establishing a bijection

Sample Problem:
(Evan Chen) Determine the number of sequences of positive integers $1 = x_0 < x_1 < \ldots < x_{10} = 10^5$ with the property that for each $m = 0, \ldots, 9$ the number $\frac{x_{m+1}}{x_m}$ is a prime number.

Chapter 7: Stars and Bars

- Using the stars and bars technique to solve a variety of counting problems

Sample Problem:

Chapter 8: Binomial

- Binomial theorem, Pascal's triangle, Sierpinski's triangle
- Various combinatorial identities, such as the hockey stick identity

Sample Problem:
(AlphaStar) What is the coefficient of x^{42} in the expansion of $(1 + x + x^2 + \cdots + x^{39})(1 + x + \cdots + x^{40})(1 + x + \cdots + x^{41})$?

Chapter 9: Counting with Recursion

- Solving counting problems by setting up a recursion and/or finding patterns

Sample Problem:
(Ali Gurel) Gustavo is filling his 2×10 room completely with 10 carpets of size 1×2. In how many different ways can he do this?

Chapter 10: Probability-1

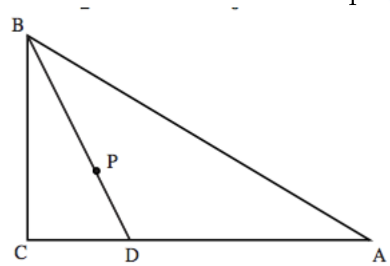
- Basic probability definitions and axioms
- Definitions of complementary events, independence, disjoint events

Sample Problem: (AIME-2002-I-1) Many states use a sequence of three letters followed by a sequence of three digits as their standard license-plate pattern. Given that each three-letter three-digit arrangement is equally likely, the probability that such a license plate will contain at least one palindrome (a three-letter arrangement or a three-digit arrangement that reads the same left-to-right as it does right-to-left) is m/n, where m and n are relatively prime positive integers. Find $m + n$.

Chapter 11: Probability-2

- Conditional probability, Bayes' theorem
- Geometric probability

Sample Problem:
(AMC12-2002-A22) Triangle ABC is a right triangle with $\angle ACB$ as its right angle, $m\angle ABC = 60°$, and $AB = 10$. Let P be randomly chosen inside $\triangle ABC$, and extend $\overline{BP}$ to meet $\overline{AC}$ at D. What is the probability that $BD > 5\sqrt{2}$?



(A) $\frac{2-\sqrt{2}}{2}$ **(B)** $\frac{1}{3}$ **(C)** $\frac{3-\sqrt{3}}{3}$ **(D)** $\frac{1}{2}$ **(E)** $\frac{5-\sqrt{5}}{5}$

Chapter 12: Expected Value

- Expected value and linearity of expectation (for an arbitrary number of events)
- Introduction to state diagrams, Markov chains

Sample Problem:
(AMC12-2016-B19) Tom, Dick, and Harry are playing a game. Starting at the same time, each of them flips a fair coin repeatedly until he gets his first head, at which

point he stops. What is the probability that all three flip their coins the same number of times?

(A) $\frac{1}{8}$ **(B)** $\frac{1}{7}$ **(C)** $\frac{1}{6}$ **(D)** $\frac{1}{4}$ **(E)** $\frac{1}{3}$

MC30G
AMC 10/12 Basic Geometry

Chapter 1: Angles

- Angles (review)
- Inscribed angles in a circle, cyclic quadrilaterals

Sample Problem:
(Alec Sun) In a regular 9-gon $ABCDEFGHI$, draw a circle that is tangent to IA at A and CD at C. What is the degree measure of minor arc AC?

Chapter 2: Special Triangles

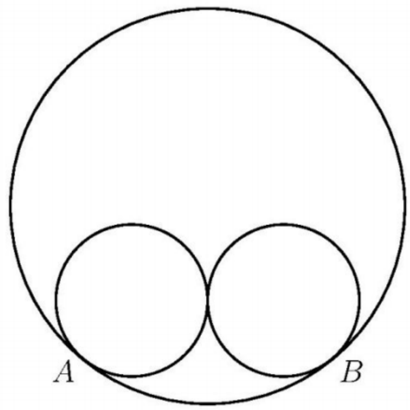
- 30-60-90, 45-45-90, and 15-75-90 triangles
- Pythagorean triples

Sample Problem:
(Math Day at the Beach-2012-Team-1) Each of two congruent equilateral triangles with side s has center that is a vertex of the other triangle. What is the area of the overlap, in terms of s?

Chapter 3: Similarity

- Similarity/congruence axioms (SSS, SAS, ASA, AA similarity)
- Power of a point, angle bisector theorem

Sample Problem:
(AMC10-2018-A15) Two circles of radius 5 are externally tangent to each other and are internally tangent to a circle of radius 13 at points A and B, as shown in the diagram. The distance AB can be written in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. What is $m+n$?

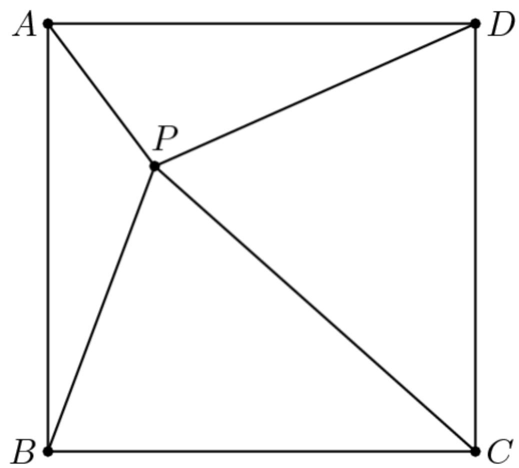


(A) 21 **(B)** 29 **(C)** 58 **(D)** 69 **(E)** 93

Chapter 4: Special Points

- Special points of a triangle (centroid, incenter, circumcenter, orthocenter)

Sample Problem:
(AMC12-2018-B13) Square $ABCD$ has side length 30. Point P lies inside the square so that $AP = 12$ and $BP = 26$. The centroids of $\triangle ABP, \triangle BCP, \triangle CDP$, and $\triangle DAP$ are the vertices of a convex quadrilateral. What is the area of that quadrilateral?



(A) $100\sqrt{2}$ **(B)** $100\sqrt{3}$ **(C)** 200 **(D)** $200\sqrt{2}$ **(E)** $200\sqrt{3}$

Chapter 5: Length-1

- Triangle inequality, Ravi substitution
- Pythagorean theorem, distance formula
- Stewart's theorem

Sample Problem:
(AMC12-2012-A12) A square region $ABCD$ is externally tangent to the circle with equation $x^2 + y^2 = 1$ at the point $(0, 1)$ on the side CD. Vertices A and B are on the circle with equation $x^2 + y^2 = 4$. What is the side length of the square?

(A) $\frac{\sqrt{10}+5}{10}$ **(B)** $\frac{2\sqrt{5}}{5}$ **(C)** $\frac{2\sqrt{2}}{3}$ **(D)** $\frac{2\sqrt{19}-4}{5}$ **(E)** $\frac{9-\sqrt{17}}{5}$

Chapter 6: Length-2

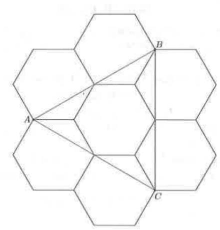
- Mass points considering levers/torque
- Ceva's theorem, Menelaus' theorem

Sample Problem:
(Lehigh MC-2014-27) Let BE be a median of triangle ABC, and let D be a point on AB such that $BD/DA = 3/7$. What is the ratio of the area of triangle BED to that of triangle ABC?

Chapter 7: Area-1

- Areas of simple shapes (triangle, certain quadrilaterals)
- Triangle area formulas (Heron's formula, A = rs, A = abc/4R, (ab sin C)/2

Sample Problem:
(AMC10-2014-B13) Six regular hexagons surround a regular hexagon of side length 1 as shown. What is the area of $\triangle ABC$?



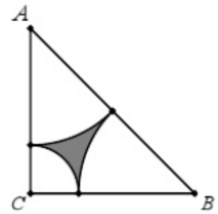
(A) $2\sqrt{3}$ **(B)** $3\sqrt{3}$ **(C)** $1 + 3\sqrt{2}$ **(D)** $2 + 2\sqrt{3}$ **(E)** $3 + 2\sqrt{3}$

Chapter 8: Area-2

- Area formula for a circle, sector
- Brahmagupta's formula
- Area of more complicated shapes involving circles and/or other polygons

Sample Problem:
(ARML-0000-Team-1) In ΔABC, $m\angle A = m\angle B = 45°$ and $AB = 16$. Mutually tangent circular arcs are drawn centered at all three vertices; the arcs centered at A and B intersect at the midpoint of $\overline{AB}$. Compute the area of the region inside the

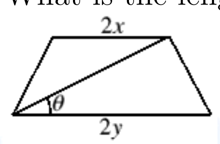
triangle and outside of the three arcs.



Chapter 9: Trigonometry-1

- Definitions of sin, cos, tan, as well as csc, sec, cot
- The unit circle
- Basic trig identities, Sum and difference formulas for sin, cos, tan (e.g. sin(a+b))

Sample Problem:
(UK MC-2010-Senior-14) The parallel sides of a trapezium have lengths $2x$ and $2y$ respectively. The diagonals are equal in length, and a diagonal makes an angle θ with the parallel sides, as shown. What is the length of each diagonal?



(A) $x+y$ **(B)** $\dfrac{x+y}{\sin\theta}$ **(C)** $(x+y)\cos\theta$ **(D)** $(x+y)\tan\theta$ **(E)** $\dfrac{x+y}{\cos\theta}$

Chapter 10: Trigonometry-2

- Law of sines
- Law of cosines
- Ratio lemma, trig Ceva's theorem
- Solving algebra problems by trig substitution

Sample Problem:
(Hong Kong MC-2006-17) If a square can completely cover a triangle with side lengths 3, 4 and 5, find the smallest possible side length of the square.

Chapter 11: Analytic Geometry

- Slope, equation of a line using slope-intercept or point-slope form, distance and midpoint formulas
- Reflections over lines in the coordinate plan
- Equation of a circle
- Shoelace formula

Sample Problem:
(AMC12-2006-B16) Regular hexagon $ABCDEF$ has vertices A and C at $(0,0)$ and $(7,1)$, respectively. What is its area?

(A) $20\sqrt{3}$ **(B)** $22\sqrt{3}$ **(C)** $25\sqrt{3}$ **(D)** $27\sqrt{3}$ **(E)** 50

Chapter 12: 3D

- Distance formula in 3D
- Area/volume of various 3D shapes (cube, prisms, cylinders, cones, spheres)
- Common 3D solids
- Euler's polyhedral formula

Sample Problem:
(HMMT Feb-2004-Guts-18) On a spherical planet with diameter 10,000 km, powerful explosives are placed at the north and south poles. The explosives are designed to vaporize all matter within 5,000 km of ground zero and leave anything beyond 5,000 km untouched. After the explosives are set off, what is the new surface area of the planet, in square kilometers?

MC30N
AMC 10/12 Basic Number Theory

Chapter 1: Gauss Sums

- Sums of arithmetic sequences
- Sum of squares and sum of cubes formula
- Sigma notation

Sample Problem: (Australian MC-2000-I22) The number 2000 is expressed as the sum of 32 consecutive positive integers. The largest of these integers is

(A) 33 **(B)** 42 **(C)** 77 **(D)** 78 **(E)** 79

Chapter 2: Primes & Prime Factorization

- Definition of divisibility $(a \mid b)$
- Prime and composite numbers, Euclid's proof of the infinitude of primes
- Fundamental theorem of arithmetic
- Legendre's formula
- Prime number theorem, prime checking algorithms (optional)

Sample Problem:
(UK MC-2008-Senior-19) How many prime numbers p are there such that $199p + 1$ is a perfect square?

(A) 0 **(B)** 1 **(C)** 2 **(D)** 4 **(E)** 8

Chapter 3: Divisibility Rules

- Divisibility by numbers 2-11 inclusive
- Various problems involving divisibility, prime factorization, etc.

Sample Problem:
(Richard Spence) Find the largest positive integer N satisfying the following properties:

- The digits of N are all odd and distinct
- N is a multiple of 11

Chapter 4: Number & Sum of Divisors

- General formula for the number and sum of divisors of a positive integer n, given its prime factorization
- Perfect, abundant, and deficient numbers (optional)

Sample Problem:
(Lehigh MC-2006-32) Let S denote the set of all (positive) divisors of 60^5. The product of all the numbers in S equals 60^e for some integer e. What is the value of e?

Chapter 5: Factoring Techniques

- Difference of squares
- Sum and difference of cubes; sum and difference of n-th powers
- Simon's Favorite Factoring Trick
- Sophie Germain identity
- Informal definition of an irreducible polynomial over the integers (e.g. $x^2 + y^2$)

Sample Problem:
(Lehigh MC-2004-37) How many ordered pairs (x, y) of integers satisfy $\frac{1}{x} + \frac{1}{y} = \frac{1}{2}$? (Note that both positive and negative integers are allowed.)

Chapter 6: Number Bases

- Conversion between different number bases (emphasis on base 2, 8, 10, and 16)
- Arithmetic in different bases
- Fast base conversion (e.g. binary to hexadecimal)

Sample Problem:
(AMC12-2012-B11) In the equation below, A and B are consecutive positive integers, and A, B, and $A+B$ represent number bases:

$$132_A + 43_B = 69_{A+B}$$

What is $A+B$?

(A) 9 **(B)** 11 **(C)** 13 **(D)** 15 **(E)** 17

Chapter 7: GCD & LCM

- Definition of relatively prime
- Computing the GCD and LCM using the prime factorization
- Computing the GCD of two numbers using the Euclidean algorithm

Sample Problem: (BMT-2013-Individual-5) Two positive integers m and n satisfy

$$\max(m,n) = (m-n)^2$$
$$\gcd(m,n) = \frac{\min(m,n)}{6}$$

Find $\operatorname{lcm}(m,n)$.

Chapter 8: Modular Arithmetic

- Basic properties of the modulo (reflexive, symmetric, transitive, etc.)
- Proof of divisibility rules using modular arithmetic

- Modular inverses
- More advanced modulo calculations involving basic operations

Sample Problem:
(Metehan Ozsoy) What is the remainder when $(1 + 2 + \cdots + 49)^{49}$ is divided by 50?

Chapter 9: Fermat's Little Theorem

- Definition of reduced residue systems $\pmod{m}$
- Applying Fermat's little theorem to find the remainder when a power is divided by a prime

Sample Problem:
(BMT-2019-Team-2) Find the remainder when 2^{2019} is divided by 7.

Chapter 10: Euler Theorem

- Definition of the totient function $\phi(n)$
- Using the totient function on basic problems involving relatively prime integers
- Definition of Euler's totient theorem
- Demonstrating that Fermat's little theorem is a special case of Euler's totient theorem

Sample Problem:
(Richard Spence) What is the sum of all positive integers less than 6! which are relatively prime to 6!?

Chapter 11: Chinese Remainder Theorem

- Applying the Chinese remainder theorem to more advanced modular arithmetic problems
- Directly computing solutions to systems of congruences

- Using the Chinese remainder theorem backwards

Sample Problem:
(Richard Spence) Find all 3-digit positive integers N such that the numbers N, $N+1$, and $N+2$ are divisible by 7, 8, and 9 respectively.

Chapter 12: Diophantine Equations

- Solving linear Diophantine equations of the form $ax + by = c$
- Bézout's identity, using the reverse Euclidean Algorithm
- Chicken McNugget theorem
- Finding Pythagorean triples
- Using modular arithmetic to solve Diophantine equations, or to show there is no integer solution

Sample Problem:
(AIME-2012-II-1) Find the number of ordered pairs of positive integer solutions (m, n) to the equation $20m + 12n = 2012$.

MC35A
AMC 10/12 Advanced Algebra

Chapter 1: Arithmetic

- Word problems using arithmetic with integers, fractions, decimals, and percent
- Decimals with repeating/terminating digits
- Rational/Irrational numbers

Sample Problem:
(AMC12-2002-A20) Suppose that a and b are digits, not both nine and not both zero, and the repeating decimal $0.\overline{ab}$ is expressed as a fraction in lowest terms. How many different denominators are possible?

(A) 3 **(B)** 4 **(C)** 5 **(D)** 8 **(E)** 9

Chapter 2: Exponents & Radicals

- Properties of exponents and radicals
- Negative/fractional exponents
- Rationalizing the denominator, simplifying radicals
- Using conjugates of radicals

Sample Problem:
(Lehigh MC-2016-34) What is the smallest integer larger than $(\sqrt{5}+\sqrt{3})^6$

Chapter 3: Word Problems & System of Equations

- Word problems, systems of equations in two or more variables

Sample Problem:
(AMC10-2002-B20) Let a, b, and c be real numbers such that $a - 7b + 8c = 4$ and $8a + 4b - c = 7$. Then $a^2 - b^2 + c^2$ is

(A) 0 **(B)** 1 **(C)** 4 **(D)** 7 **(E)** 8

Chapter 4: Time, Travel, Work

- Distance = Rate × Time, average speed
- Harmonic mean
- Relative speed
- Rate/Work Problems

Sample Problem:
(AMC10-2012-A19) Paula the painter and her two helpers each paint at constant, but different, rates. They always start at 8:00 AM, and all three always take the same amount of time to eat lunch. On Monday the three of them painted 50% of a house, quitting at 4:00 PM. On Tuesday, when Paula wasn't there, the two helpers painted only 24% of the house and quit at 2:12 PM. On Wednesday Paula worked by herself and finished the house by working until 7:12 P.M. How long, in minutes, was each day's lunch break?

(A) 30 **(B)** 36 **(C)** 42 **(D)** 48 **(E)** 60

Chapter 5: Sequences-1

- Mean, median, mode, range
- Arithmetic and geometric sequences
- Geometric series formula and derivation

Sample Problem:
(AMC10-2000-A23) When the mean, median, and mode of the list $10, 2, 5, 2, 4, 2, x$ are arranged in increasing order, they form a non-constant arithmetic progression. What is the sum of all possible real values of x?

(A) 3 **(B)** 6 **(C)** 9 **(D)** 17 **(E)** 20

Chapter 6: Sequences-2

- Recurrent sequences
- Finding the general term via patterns

Sample Problem:
(Lehigh MC-2008-33) A Fibonacci-like sequence of numbers is defined by $a_1 = 1$, $a_2 = 3$, and for $n \geq 3$, $a_n = a_{n-1} + a_{n-2}$. One can compute that $a_{29} = 1149851$ and $a_{30} = 1860498$. What is the value of $\sum_{n=1}^{28} a_n$?

Chapter 7: Functions & Operations

- Definitions of function, domain, codomain/range
- Injective, surjective, bijective functions
- Inverse functions
- Operators
- Simple functional equations

Sample Problem:
(Lehigh MC-2002-36) If $2f(x) + f(1-x) = x^2$ for all x, then $f(x) =$

Chapter 8: Polynomials-1

- Polynomials of a single variable; definitions of degree, root, etc.
- Solving for the roots of a quadratic by factoring, completing the square, or quadratic formula

- Rational root theorem
- Fundamental theorem of algebra
- Less emphasis on complex numbers (Chapter 12)

Sample Problem:
(Aaron Lin, David Zhu) Suppose P is a monic quartic polynomial (i.e. a 4th-degree polynomial with leading coefficient 1) such that $P(1) = 1$, $P(2) = 4$, $P(3) = 9$, $P(4) = 16$. Find $P(5)$.

Chapter 9: Polynomials-2

- Generalized Vieta's formulas
- Manipulation of symmetric sums to produce other expressions

Sample Problem:
(HMMT Nov-2016-Guts-27) Let r_1, r_2, r_3, r_4 be the four roots of the polynomial $x^4 - 4x^3 + 8x^2 - 7x + 3$. Find the value of

$$\frac{r_1^2}{r_2^2 + r_3^2 + r_4^2} + \frac{r_2^2}{r_1^2 + r_3^2 + r_4^2} + \frac{r_3^2}{r_1^2 + r_2^2 + r_4^2} + \frac{r_4^2}{r_1^2 + r_2^2 + r_3^2}.$$

Chapter 10: Trigonometry

- Review of trigonometric functions (sin, cos, tan, csc, sec, cot)
- More emphasis on trigonometric identities (addition and multiple-angle formulae)
- Solving algebra problems via trig substitution

Sample Problem:
(BMT-2016-Analysis-5) Find

$$\frac{\tan 1^\circ}{1 + \tan 1^\circ} + \frac{\tan 2^\circ}{1 + \tan 2^\circ} + \cdots + \frac{\tan 89^\circ}{1 + \tan 89^\circ}.$$

Chapter 11: Logarithm

- Definition of a logarithm in base b, simple logarithmic identities (change-of-base formula, addition/subtraction of logarithms)
- Natural logarithms, the number e
- Applications: Binary search, merge sort example

Sample Problem:
(ARML-2010-Team-2) Define $\log^*(n)$ to be the smallest number of times the log function must be iteratively applied to n to get a result less than or equal to 1. For example, $\log^*(1000) = 2$ since $\log 1000 = 3$ and $\log(\log 1000) = \log 3 = 0.477\ldots \leq 1$. Let a be the smallest integer such that $\log^*(a) = 3$. Compute the number of zeros in the base 10 representation of a.

Chapter 12: Complex Numbers

- More rigorous introduction to complex numbers
- Review of the Fundamental Theorem of Algebra, conjugate root theorem
- Polar form, Euler's formula, de Moivre's formula
- Roots of unity

Sample Problem:
(Victor Chen) If $z = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$, compute $z^{100} + \frac{1}{z^{100}}$.

MC35C
AMC 10/12 Advanced Counting

Chapter 1: Counting Basics

- Addition/multiplication principles
- Permutations, combinations, binomial coefficients

Sample Problem:
(SMT-2018-General-15) How many ways are there to select distinct integers x, y, where $1 \leq x \leq 25$ and $1 \leq y \leq 25$, such that $x + y$ is divisible by 5?

Chapter 2: Casework

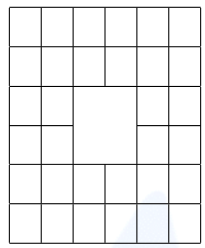
- Solving a variety of counting problems using casework
- Use casework to break difficult problems into easier pieces

Sample Problem:
(HMMT Feb-2006-Combinatorics-6) For how many ordered triplets (a, b, c) of positive integers less than 10 is the product $a \times b \times c$ divisible by 20?

Chapter 3: Complementary Counting & Overcounting

- Solving counting problems using the techniques of complementary counting and/or overcounting

Sample Problem:
(HMMT Feb-2008-Guts-6) Determine the number of non-degenerate rectangles whose edges lie completely on the grid lines of the following figure.



Chapter 4: Counting Sets

- Definitions of set, cardinality, union, intersection
- Principle of inclusion-exclusion for two or more sets

Sample Problem:
(HMMT Feb-2010-Combinatorics-1) Let $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. How many (potentially empty) subsets T of S are there such that, for all x, if x is in T and $2x$ is in S then $2x$ is also in T?

Chapter 5: Counting with Digits

- Solving a variety of counting problems involving digits of a number
- Counting palindromes

Sample Problem:
(AMC12-2008-A21) A permutation $(a_1, a_2, a_3, a_4, a_5)$ of $(1, 2, 3, 4, 5)$ is heavy-tailed if $a_1 + a_2 < a_4 + a_5$. What is the number of heavy-tailed permutations?

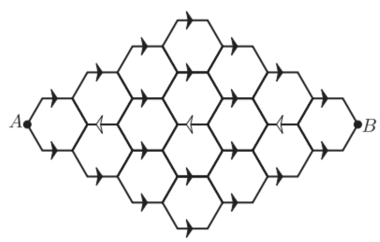
(A) 36 **(B)** 40 **(C)** 44 **(D)** 48 **(E)** 52

Chapter 6: Path Counting & Bijections

- Definitions of injective, surjective, and bijective functions

- Examples of bijections between two infinite sets (e.g. the set of whole numbers and the set of integers)
- Solving counting problems by establishing a bijection

Sample Problem:
(AMC10-2012-B25) A bug travels from A to B along the segments in the hexagonal lattice pictured below. The segments marked with an arrow can be traveled only in the direction of the arrow, and the bug never travels the same segment more than once. How many different paths are there?



(A) 2112 **(B)** 2304 **(C)** 2368 **(D)** 2384 **(E)** 2400

Chapter 7: Stars and Bars

- Using the stars and bars technique to solve a variety of counting problems

Sample Problem:
(Caleb Ji) How many ways can David pick four of the first twelve positive integers such that no two of the numbers he picks are consecutive?

Chapter 8: Binomial

- Binomial theorem, Pascal's triangle, Sierpinski's triangle
- Various combinatorial identities, such as the hockey stick identity

Sample Problem:
(AMC10-2011-B23) What is the hundreds digit of 2011^{2011}?

(A) 1 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 9

Chapter 9: Counting with Recursion

- Solving counting problems by setting up a recursion and/or finding patterns

Sample Problem:
(Lehigh MC-2014-26) How many 10-digit strings of 0's and 1's are there that do not contain any consecutive 0's?

Chapter 10: Probability-1

- Basic probability definitions and axioms
- Definitions of complementary events, independence, disjoint events

Sample Problem:
(AMC10-2004-B23) Each face of a cube is painted either red or blue, each with probability $\frac{1}{2}$. The color of each face is determined independently. What is the probability that the painted cube can be placed on a horizontal surface so that the four vertical faces are all the same color?

(A) $\frac{1}{4}$ **(B)** $\frac{5}{16}$ **(C)** $\frac{3}{8}$ **(D)** $\frac{7}{16}$ **(E)** $\frac{1}{2}$

Chapter 11: Probability-2

- Conditional probability, Bayes' theorem
- Geometric probability

Sample Problem:
(AMC10-2012-A25) Real numbers x, y, and z are chosen independently and at random from the interval $[0, n]$ for some positive integer n. The probability that no two of x, y, and z are within 1 unit of each other is greater than $\frac{1}{2}$. What is the smallest possible value of n?

(A) 7 **(B)** 8 **(C)** 9 **(D)** 10 **(E)** 11

Chapter 12: Expected Value

- Expected value and linearity of expectation (for an arbitrary number of events)
- Introduction to state diagrams, Markov chains

Sample Problem:
(HMMT Nov-2010-General1-4) An ant starts at the point $(1, 0)$. Each minute, it walks from its current position to one of the four adjacent lattice points until it reaches a point (x, y) with $|x| + |y| \geq 2$. What is the probability that the ant ends at the point $(1, 1)$?

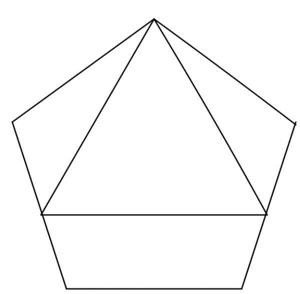
MC35G
AMC 10/12 Advanced Geometry

Chapter 1: Angles

- Angles (review)
- Inscribed angles in a circle, cyclic quadrilaterals

Sample Problem:
(Math Day at the Beach-2012-Individual-19) The figure below contains a regular pentagon and an equilateral triangle. Let $a < b < c < d < e$ be all the different measures of all of the angles in the picture. Compute

$$\frac{b}{a} + \frac{e}{d} + \frac{d}{b}.$$



Chapter 2: Special Triangles

- 30-60-90, 45-45-90, and 15-75-90 triangles

- Pythagorean triples

Sample Problem:
(BMT-2016-Geometry-4) ABC is an equilateral triangle, and $ADEF$ is a square. If D lies on side AB and E lies on side BC, what is the ratio of the area of the equilateral triangle to the area of the square?

Chapter 3: Similarity

- Similarity/congruence axioms (SSS, SAS, ASA, AA similarity)
- Power of a point, angle bisector theorem

Sample Problem:
(AMC10-2018-A24) Triangle ABC with $AB = 50$ and $AC = 10$ has area 120. Let D be the midpoint of $\overline{AB}$, and let E be the midpoint $\overline{AC}$. The angle bisector of $\angle BAC$ intersects $\overline{DE}$ and $\overline{BC}$ at F and G, respectively. What is the area of quadrilateral $FDBG$?

(A) 60 **(B)** 65 **(C)** 70 **(D)** 75 **(E)** 80

Chapter 4: Special Points

- Special points of a triangle (centroid, incenter, circumcenter, orthocenter)

Sample Problem:
(AMC12-2012-A18) Triangle ABC has $AB = 27, AC = 26$, and $BC = 25$. Let I denote the intersection of the internal angle bisectors $\triangle ABC$. What is BI?

(A) 15 **(B)** $5 + \sqrt{26} + 3\sqrt{3}$ **(C)** $3\sqrt{26}$ **(D)** $\frac{2}{3}\sqrt{546}$ **(E)** $9\sqrt{3}$

Chapter 5: Length-1

- Triangle inequality, Ravi substitution
- Pythagorean theorem, distance formula
- Stewart's theorem

Sample Problem:
(AMC10-2004-B22) A triangle with sides of 5,12, and 13 has both an inscribed and a circumscribed circle. What is the distance between the centers of those circles?

(A) $\frac{3\sqrt{5}}{2}$ **(B)** $\frac{7}{2}$ **(C)** $\sqrt{15}$ **(D)** $\frac{\sqrt{65}}{2}$ **(E)** $\frac{9}{2}$

Chapter 6: Length-2

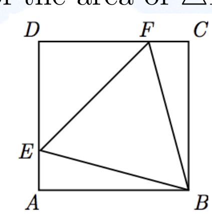
- Mass points considering levers/torque
- Ceva's theorem, Menelaus' theorem

Sample Problem:
(Challenging Pr in Geo p37 q8) In right $\triangle ABC$, P and Q are on $\overline{BC}$ and $\overline{AC}$, respectively, such that $CP = CQ = 2$. Through the point of intersection, R, of $\overline{AP}$ and $\overline{BQ}$, a line is drawn also passing through C and meeting $\overline{AB}$ at S. $\overline{PQ}$ extended meets line AB at T. If the hypotenuse $AB = 10$ and $AC = 8$, find TS.

Chapter 7: Area-1

- Areas of simple shapes (triangle, certain quadrilaterals)
- Triangle area formulas (Heron's formula, A = rs, A = abc/4R, (ab sin C)/2

Sample Problem:
(AMC10-2004-A20) Points E and F are located on square $ABCD$ so that $\triangle BEF$ is equilateral. What is the ratio of the area of $\triangle DEF$ to that of $\triangle ABE$?



(A) $\frac{4}{3}$ **(B)** $\frac{3}{2}$ **(C)** $\sqrt{3}$ **(D)** 2 **(E)** $1+\sqrt{3}$

Chapter 8: Area-2

- Area formula for a circle, sector
- Brahmagupta's formula
- Area of more complicated shapes involving circles and/or other polygons

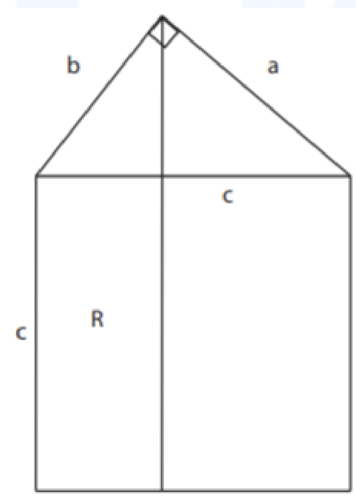
Sample Problem:
(AMC10-2002-A19) Spot's doghouse has a regular hexagonal base that measures one yard on each side. He is tethered to a vertex with a two-yard rope. What is the area, in square yards, of the region outside the doghouse that Spot can reach?

(A) $\frac{2}{3}\pi$ **(B)** 2π **(C)** $\frac{5}{2}\pi$ **(D)** $\frac{8}{3}\pi$ **(E)** 3π

Chapter 9: Trigonometry-1

- Definitions of sin, cos, tan, as well as csc, sec, cot
- The unit circle
- Basic trig identities, Sum and difference formulas for sin, cos, tan (e.g. sin(a+b))

Sample Problem:
(BMT-2012-Tournament-Round1-P5) The legs of the right triangle shown below have length $a = 255$ and $b = 32$. Find the area of the smaller rectangle (the one labeled R).



Chapter 10: Trigonometry-2

- Law of sines
- Law of cosines
- Ratio lemma, trig Ceva's theorem
- Solving algebra problems by trig substitution

Sample Problem:
(HMMT Nov-2010-Guts-17) A triangle with side lengths $5, 7, 8$ is inscribed in a circle C. The diameters of C parallel to the sides of lengths 5 and 8 divide C into four sectors. What is the area of either of the two smaller ones?

Chapter 11: Analytic Geometry

- Slope, equation of a line using slope-intercept or point-slope form, distance and midpoint formulas
- Reflections over lines in the coordinate plan
- Equation of a circle
- Shoelace formula

Sample Problem:
(AMC10-2016-B21) What is the area of the region enclosed by the graph of the equation $x^2 + y^2 = |x| + |y|$?

(A) $\pi + \sqrt{2}$ **(B)** $\pi + 2$ **(C)** $\pi + 2\sqrt{2}$ **(D)** $2\pi + \sqrt{2}$ **(E)** $2\pi + 2\sqrt{2}$

Chapter 12: 3D

- Distance formula in 3D
- Area/volume of various 3D shapes (cube, prisms, cylinders, cones, spheres)
- Common 3D solids
- Euler's polyhedral formula

Sample Problem:

(HMMT Nov-2010-Guts-17) A triangle with side lengths $5, 7, 8$ is inscribed in a circle C. The diameters of C parallel to the sides of lengths 5 and 8 divide C into four sectors. What is the area of either of the two smaller ones?

MC35N
AMC 10/12 Advanced Number Theory

Chapter 1: Gauss Sums

- Sums of arithmetic sequences
- Sum of squares and sum of cubes formula
- Sigma notation

Sample Problem:
(HMMT Nov-2013-Guts-9) Find the remainder when $1^2 + 3^2 + 5^2 + \ldots + 99^2$ is divided by 1000.

Chapter 2: Primes & Prime Factorization

- Definition of divisibility ($a \mid b$)
- Prime and composite numbers, Euclid's proof of the infinitude of primes
- Fundamental theorem of arithmetic
- Legendre's formula
- Prime number theorem, prime checking algorithms (optional)

Sample Problem:
(AIME-2006-II-3) Let P be the product of the first 100 positive odd integers. Find the largest integer k such that P is divisible by 3^k.

Chapter 3: Divisibility Rules

- Divisibility by numbers 2-11 inclusive
- Various problems involving divisibility, prime factorization, etc.

Sample Problem:
(Mehmet Kaysi) The number $\overline{406828a}$, where a is a digit, is an odd perfect square which is not a multiple of 9. What is the digit a?

Chapter 4: Number & Sum of Divisors

- General formula for the number and sum of divisors of a positive integer n, given its prime factorization
- Perfect, abundant, and deficient numbers (optional)

Sample Problem:
(ARML-2014-Individual-6) Compute the smallest positive integer n such that $214 \cdot n$ and $2014 \cdot n$ have the same number of divisors.

Chapter 5: Factoring Techniques

- Difference of squares
- Sum and difference of cubes; sum and difference of n-th powers
- Simon's Favorite Factoring Trick
- Sophie Germain identity
- Informal definition of an irreducible polynomial over the integers (e.g. x^2+y^2)

Sample Problem:
(SMT-2018-General-18) How many integer pairs (a, b) satisfy $\frac{1}{a} + \frac{1}{b} = \frac{1}{2018}$?

Chapter 6: Number Bases

- Conversion between different number bases (emphasis on base 2, 8, 10, and 16)
- Arithmetic in different bases
- Fast base conversion (e.g. binary to hexadecimal)

Sample Problem:
(SMT-2012-Advanced Topics-2) Find the sum of all integers $x, x \geq 3$, such that

$$201020112012_x$$

(that is, 201020112012 interpreted as a base x number) is divisible by $x - 1$.

Chapter 7: GCD & LCM

- Definition of relatively prime
- Computing the GCD and LCM using the prime factorization
- Computing the GCD of two numbers using the Euclidean algorithm

Sample Problem:
(HMMT Nov-2015-Guts-15) Find the smallest positive integer b such that 1111_b (1111 in base b) is a perfect square. If no such b exists, write "No solution."

Chapter 8: Modular Arithmetic

- Basic properties of the modulo (reflexive, symmetric, transitive, etc.)
- Proof of divisibility rules using modular arithmetic
- Modular inverses
- More advanced modulo calculations involving basic operations

Sample Problem:
(AIME-2010-I-2) Find the remainder when $9 \cdot 99 \cdot 999 \cdot \cdots \cdot \underbrace{99 \cdots 9}_{\text{999 9's}}$ is divided by 1000.

Chapter 9: Fermat's Little Theorem

- Definition of reduced residue systems $\pmod m$
- Applying Fermat's little theorem to find the remainder when a power is divided by a prime

Sample Problem: (SMT-2019-Discrete-1) How many nonnegative integers less than 2019 are not solutions to $x^8 + 4x^6 - x^2 + 3 \equiv 0 \pmod 7$?

Chapter 10: Euler Theorem

- Definition of the totient function $\phi(n)$
- Using the totient function on basic problems involving relatively prime integers
- Definition of Euler's totient theorem
- Demonstrating that Fermat's little theorem is a special case of Euler's totient theorem

Sample Problem:
(Ata Pir) Find the smallest integer n, such that $\frac{\varphi(n)}{n} < \frac{1}{4}$.

Chapter 11: Chinese Remainder Theorem

- Applying the Chinese remainder theorem to more advanced modular arithmetic problems
- Directly computing solutions to systems of congruences
- Using the Chinese remainder theorem backwards

Sample Problem:
(AMC10-2010-A24) The number obtained from the last two nonzero digits of 90! is equal to n. What is n?

(A) 12 **(B)** 32 **(C)** 48 **(D)** 52 **(E)** 68

Chapter 12: Diophantine Equations

- Solving linear Diophantine equations of the form $ax + by = c$
- Bézout's identity, using the reverse Euclidean Algorithm
- Chicken McNugget theorem
- Finding Pythagorean triples
- Using modular arithmetic to solve Diophantine equations, or to show there is no integer solution

Sample Problem:
(CHMMC-2010 Winter-Individual-9) Let A and B be points in the plane such that $AB = 30$. A circle with integer radius passes through A and B. A point C is constructed on the circle such that $\overline{AC}$ is a diameter of the circle. Compute all possible radii of the circle such that BC is a positive integer.

MC40A
AIME Basic Algebra

Chapter 1: Word Problems

- Developing logical analysis and boost creative thinking by solving word problems.
- Converting word problems into mathematical equations and solving AIME level system of equations.

Sample Problem:
(PUMaC-2012-Team-2.3.1) For some reason, people in math problems like to paint houses. Alice can paint a house in one hour. Bob can paint a house in six hours. If they work together, it takes them seven hours to paint a house. You might be thinking "What? That's not right!" but I did not make a mistake.
When Alice and Bob work together, they get distracted very easily and simultaneously send text messages to each other. When they are texting, they are not getting any work done. When they are not texting, they are painting at their normal speeds (as if they were working alone). Carl, the owner of the house decides to check up on their work. He randomly picks a time during the seven hours. The probability that they are texting during that time can be written as r/s, where r and s are integers and $\gcd(r, s) = 1$. What is $r + s$?

Chapter 2: Sequences & Series

- Finding patterns in sequences by looking at small cases.
- Using trig substitution and invariance in sequence problems.

- Understanding recurrence relations and solving linear recurrences.
- Finding closed-form formulas for sequences.

Sample Problem:
(AMC12-2016-B25) The sequence (a_n) is defined recursively by $a_0 = 1, a_1 = \sqrt[19]{2}$, and $a_n = a_{n-1}a_{n-2}^2$ for $n \geq 2$. What is the smallest positive integer k such that the product $a_1 a_2 \cdots a_k$ is an integer?

(A) 17 **(B)** 18 **(C)** 19 **(D)** 20 **(E)** 21

Chapter 3: Functions-1

- Solving equations that involve special functions such as floor, ceiling and absolute value
- Counting functions using information about its domain and range

Sample Problem:
(Hong Kong MC-2010-14) Let $\lfloor x \rfloor$ denote the greatest integer not exceeding x, e.g. $\lfloor \pi \rfloor = 3$. Given $f(0) = 0$ and

$$f(n) = f\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + n - 2\left\lfloor \frac{n}{2} \right\rfloor$$

for any positive integer n. If m is a positive integer not exceeding 2010, find the greatest possible value of $f(m)$.

Chapter 4: Functions-2

- Solving functional equations using substitution, injectivity, and surjectivity, symmetry

Sample Problem:
(Vishal Arul) Define $\{x\}$ to be the fractional part of x; that is, $\{x\} = x - \lfloor x \rfloor$. Define $a \circ b = \lfloor ab^2 + a^4(3-a)\{|b|^{1/a}\} + a^2 - 2b^2 - 5a \rfloor + 6$. What is $1 \circ (2 \circ (3 \circ \cdots (99 \circ 100) \cdots))$?

Chapter 5: Polynomials-1

- Finding roots of some cubic, quartic, and higher degree polynomials using substitution, binomial theorem
- Vieta's theorem and its applications
- Using techniques such as long division, factor theorem and rational root theorem when finding roots of higher degree polynomials

Sample Problem:
(Iurie Boreico) A rectangular box has volume equal to 6, surface area equal to 30, and diagonal equal to $\sqrt{34}$. The largest dimension of the box is $a+\sqrt{b}$ where a, b are positive integers. Find $a+b$.

Chapter 6: Polynomials-2

- Solving polynomial equations using Lagrange interpolation and Finite differences

Sample Problem:
(Jafar Jafarov) If a, b, c are roots of $x^3+2x+7=0$, find

$$\frac{1}{a^2+1}+\frac{1}{b^2+1}+\frac{1}{c^2+1}$$

Express your answer as a common fraction in reduced form.

Chapter 7: Logarithm

- Solving AIME level problems involving logarithms and natural logarithm

Sample Problem:
(CHMMC-2010 Fall-Individual-4) Let

$$S=\log_2 9 \log_3 16 \log_4 25 \cdots \log_{999} 1000000.$$

Compute the greatest integer less than or equal to $\log_2 S$.

Chapter 8: Trigonometry

- Solving algebra problems using trig substitution, trig identities and formulas

Sample Problem:
(AMC12-2008-A25) A sequence $(a_1, b_1), (a_2, b_2), (a_3, b_3), \ldots$ of points in the coordinate plane satisfies

$$(a_{n+1}, b_{n+1}) = (\sqrt{3}a_n - b_n, \sqrt{3}b_n + a_n) \text{ for } n = 1, 2, 3, \ldots$$

Suppose that $(a_{100}, b_{100}) = (2, 4)$. What is $a_1 + b_1$?

(A) $-\frac{1}{2^{97}}$ **(B)** $-\frac{1}{2^{99}}$ **(C)** 0 **(D)** $\frac{1}{2^{98}}$ **(E)** $\frac{1}{2^{96}}$

Chapter 9: Complex Numbers-1

- Having a deep knowledge of complex numbers, finding roots of polynomials with complex roots
- Algebraic operations involving complex numbers and complex plane
- Problem solving techniques using Euler's formula and de Moivre's formula

Sample Problem:
(AIME-2016-I-7) For integers a and b consider the complex number

$$\frac{\sqrt{ab + 2016}}{ab + 100} - \left(\frac{\sqrt{|a + b|}}{ab + 100}\right) i$$

Find the number of ordered pairs of integers (a, b) such that this complex number is a real number.

Chapter 10: Complex Numbers-2

- Finding roots of unity and using algebraic operations on roots of unity to solve problems

Sample Problem:
(HMMT Feb-2008-Algebra-6) A root of unity is a complex number that is a solution to $z^n = 1$ for some positive integer n. Determine the number of roots of unity that are also roots of $z^2 + az + b = 0$ for some integers a and b.

Chapter 11: System of Equations

- Solving system of equations using polynomials, substitutions and symmetry

Sample Problem:
(AIME-2000-I-7) Suppose that x, y, and z are three positive numbers that satisfy the equations $xyz = 1$, $x + \frac{1}{z} = 5$, and $y + \frac{1}{x} = 29$. Then $z + \frac{1}{y} = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Chapter 12: Inequalities

- Finding minimum/maximum of algebraic expressions using elementary properties of inequalities, such as transitivity and algebraic operations on inequalities
- Arithmetic Mean - Geometric Mean (AM-GM) Inequality
- Cauchy-Schwarz Inequality
- Some advanced inequalities such as Rearrangement Inequality, Jensen's Inequality and weighted AM-GM Inequality

Sample Problem:
(SMT-2018-Algebra Tiebreaker-1) If a, b, c are real numbers with $a - b = 4$, find the maximum value of $ac + bc - c^2 - ab$.

MC40C
AIME Basic Counting

Chapter 1: Basic Counting Techniques

- Solving counting problems using techniques such as casework and complementary counting

Sample Problem:
(AIME-2008-I-7) Let S_i be the set of all integers n such that $100i \leq n < 100(i+1)$. For example, S_4 is the set $400, 401, 402, \ldots, 499$. How many of the sets $S_0, S_1, S_2, \ldots, S_{999}$ do not contain a perfect square?

Chapter 2: Counting Sets & PIE

- Solving counting problems using the Principle of Inclusion and Exclusion (PIE)

Sample Problem:
(Kevin Liu) How many functions $f : \{1, 2, \ldots, 6\} \to \{1, 2, \ldots, 6\}$ are there such that $\{1, 2, 3\}$ is a subset of the range of f?

Chapter 3: Path Counting & Bijections

- Solving counting problems using bijections
- Solving path-counting problems

Sample Problem:
(Brice Huang) How many ways are there to write 10 as the sum of any number of positive integers if different orderings of the same sum are distinguishable?

Chapter 4: Stars and Bars

- Solving counting problems using the Stars and Bars method

Sample Problem:
(PUMaC-2014-Team-5) How many sets of positive integers (a, b, c) satisfies $a > b > c > 0$ and $a + b + c = 103$?

Chapter 5: Binomial

- Solving counting problems involving binomials and multinomials
- Binomial identities such as Hockey-Stick Identity and Vandermonde's Identity

Sample Problem:
(AIME-2000-II-7) Given that

$$\frac{1}{2!17!} + \frac{1}{3!16!} + \frac{1}{4!15!} + \frac{1}{5!14!} + \frac{1}{6!13!} + \frac{1}{7!12!} + \frac{1}{8!11!} + \frac{1}{9!10!} = \frac{N}{1!18!}$$

find the greatest integer that is less than $\frac{N}{100}$.

Chapter 6: Counting with Recursion

- Identifying which counting problems can be solved using recursions
- Finding and solving recursions

Sample Problem:
(PUMaC-2014-Combinatorics-4) Amy has a 2×10 puzzle grid which she can use 1×1 and 1×2 (1 vertical, 2 horizontal) tiles to cover. How many ways can she exactly cover the grid without any tiles overlapping and without rotating the tiles?

Chapter 7: Probability

- Solving difficult probability problems
- Conditional probability and Bayes' Theorem

- Geometric probability

Sample Problem:
(AIME-2014-II-6) Charles has two six-sided dice. One of the die is fair, and the other die is biased so that it comes up six with probability $\frac{2}{3}$ and each of the other five sides has probability $\frac{1}{15}$. Charles chooses one of the two dice at random and rolls it three times. Given that the first two rolls are both sixes, the probability that the third roll will also be a six is $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p+q$.

Chapter 8: Expected Value

- Random variables, expected value and variance
- Solving geometry problems involving expected values
- Properties of expectation, such as linearity of expectation

Sample Problem:
(HMMT Feb-2010-Guts-9) Indecisive Andy starts out at the midpoint of the 1-unit-long segment $\overline{HT}$. He flips 2010 coins. On each flip, if the coin is heads, he moves halfway towards endpoint H, and if the coin is tails, he moves halfway towards endpoint T. After his 2010 moves, what is the expected distance between Andy and the midpoint of $\overline{HT}$? Express your answer in decimals.

Chapter 9: Markov Chains

- Solving problems using Markov chains and state diagrams

Sample Problem:
(PUMaC-2010-Combinatorics-4) Erick stands in the square in the 2nd row and 2nd column of a 5 by 5 chessboard. There are \$1 bills in the top left and bottom right squares and there are \$5 bills in the top right and bottom left squares, as shown below.

\$1				\$5
	E			
\$5				\$1

Every second, Erick randomly chooses a square adjacent to the one he currently stands in (that is, a square sharing an edge with the one he currently stands in) and moves to that square. When Erick reaches a square with money on it, he takes it and quits. The expected value of Erick's winnings in dollars is m/n, where m and n are relatively prime positive integers. Find $m + n$.

Chapter 10: Geometric Counting

- Solving counting problems related to geometric objects
- Euler's Formula

Sample Problem:
(ARML-2014-Team-2) A point is selected at random from the interior of a right triangle with legs of length $2\sqrt{3}$ and 4. Let p be the probability that the distance between the point and the nearest vertex is less than 2. Then p can be written in the form $a + \sqrt{b}\pi$, where a and b are rational numbers. Compute (a, b).

Chapter 11: Generating Functions

- Using generating functions to turn counting problems into algebra
- Counting number of partitions

Sample Problem:
(Christopher Shao) Find the number of solutions to $a+b+c = 4$ if $-3 \leq a \leq -1, 0 \leq b \leq 2, 3 \leq c \leq 5$ and a, b, and c are integers.

Chapter 12: Catalan Numbers

- Using Catalan numbers to solve counting problems

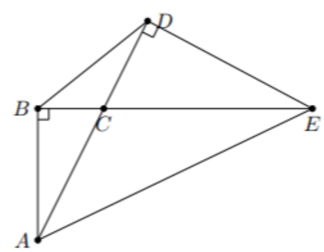
Sample Problem:
(HMMT Feb-2001-Guts-11) 12 points are placed around the circumference of a circle. How many ways are there to draw 6 non-intersecting chords joining these points in pairs?

MC40G
AIME Basic Geometry

Chapter 1: Angles

- Angles in circles and polygons; cyclic quadrilaterals
- Using angle chasing to solve problems

Sample Problem:
(CHMMC-2012 Fall-Team-4) Consider the figure below, not drawn to scale.

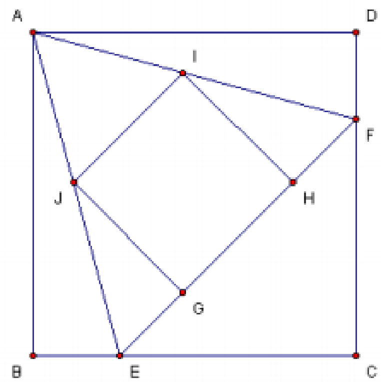


In this figure, assume that $AB \perp BE$ and $AD \perp DE$. Also, let $AB = \sqrt{6}$ and $\angle BED = \frac{\pi}{6}$. Find AC.

Chapter 2: Special Triangles

- equilateral, 30-60-90, 45-45-90, 15-75-90, 45-60-75, 36-72-72, and 18-72-90 triangles
- Pythagorean triples and Heronian scalenes

Sample Problem: (SMT-2011-Team-1) Let $ABCD$ be a unit square. The point E lies on BC and F lies on AD. $\triangle AEF$ is equilateral. $GHIJ$ is a square inscribed in $\triangle AEF$ so that GH is on EF. Compute the area of $GHIJ$.



Chapter 3: Similarity

- Similarity and congruence conditions (SSS, ASA, SAA, AA and SAS similarity, not SSA)
- Angle bisector theorem

Sample Problem:
(AMC12-2002-A23) In triangle ABC, side $\overline{AC}$ and the perpendicular bisector of $\overline{BC}$ meet in point D, and $\overline{BD}$ bisects $\angle ABC$. If $AD = 9$ and $DC = 7$, what is the area of triangle ABD?

(A) 14 **(B)** 21 **(C)** 28 **(D)** $14\sqrt{5}$ **(E)** $28\sqrt{5}$

Chapter 4: Special Points

- Properties of the four triangle centers (centroid, orthocenter, incenter, circumcenter)
- The Euler line

Sample Problem:
(AIME-2016-I-6) In $\triangle ABC$ let I be the center of the inscribed circle, and let the bisector of $\angle ACB$ intersect AB at L. The line through C and L intersects the

circumscribed circle of $\triangle ABC$ at the two points C and D. If $LI = 2$ and $LD = 3$, then $IC = \frac{p}{q}$, where p and q are relatively prime positive integers. Find $p + q$.

Chapter 5: Length-1

- Triangle inequality and Ravi substitution
- Pythagorean theorem and distance formula
- Mass points
- Ceva's Theorem, Menelaus' Theorem and Stewart's Theorem

Sample Problem:
(AIME-2008-II-5) In trapezoid $ABCD$ with $\overline{BC} \parallel \overline{AD}$, let $BC = 1000$ and $AD = 2008$. Let $\angle A = 37°$, $\angle D = 53°$, and M and N be the midpoints of $\overline{BC}$ and $\overline{AD}$, respectively. Find the length MN.

Chapter 6: Length-2

- Length problems involving circles
- Power of a point
- Radical axis and radical center
- Ptolemy's theorem

Sample Problem:
(PUMaC-2010-Geometry-6) A semicircle is folded along a chord AN and intersects its diameter MN at B. Given that $MB : BN = 2 : 3$ and $MN = 10$, if $AN = x$, find x^2.

Chapter 7: Area-1

- Triangle area formulas
- Special quadrilateral area formulas such as Brahmagupta's formula

Sample Problem:
(AIME-2016-II-7) Squares $ABCD$ and $EFGH$ have a common center and $\overline{AB}||\overline{EF}$. The area of $ABCD$ is 2016, and the area of $EFGH$ is a smaller positive integer. Square $IJKL$ is constructed so that each of its vertices lies on a side of $ABCD$ and each vertex of $EFGH$ lies on a side of $IJKL$. Find the difference between the largest and smallest positive integer values for the area of $IJKL$.

Chapter 8: Area-2

- Area problems involving length ratios

Sample Problem:
(BMT-2014-Individual-12) Suppose four coplanar points A, B, C, and D satisfy $\overline{AB} = 3, \overline{BC} = 4, \overline{CA} = 5$, and $\overline{BD} = 6$. Determine the maximal possible area of ΔACD.

Chapter 9: Trigonometry

- Definitions of trigonometric functions, basic trig identities, sum and difference formulas
- Law of sines, law of cosines, ratio lemma
- Trigonometric Ceva

Sample Problem:
(AMC12-2018-A23) In $\triangle PAT, \angle P = 36°, \angle A = 56°$, and $PA = 10$. Points U and G lie on sides $\overline{TP}$ and $\overline{TA}$, respectively, so that $PU = AG = 1$. Let M and N be the midpoints of segments $\overline{PA}$ and $\overline{UG}$, respectively. What is the degree measure of the acute angle formed by lines MN and PA?

(A) 76 **(B)** 77 **(C)** 78 **(D)** 79 **(E)** 80

Chapter 10: Analytic Geometry

- Distance formulas (between two points, point & line)
- The slope and the equation of a line (slope-intercept and point slope)

- Reflections over lines
- Equation of circles
- Shoelace formula and Pick's theorem

Sample Problem:
(AMC12-2014-A25) The parabola P has focus $(0, 0)$ and goes through the points $(4, 3)$ and $(-4, -3)$. For how many points $(x, y) \in P$ with integer coordinates is it true that $|4x + 3y| \leq 1000$?

(A) 38 **(B)** 40 **(C)** 42 **(D)** 44 **(E)** 46

Chapter 11: Complex Numbers

- Introduction to radians, Euler's formula
- Various representations of complex numbers
- Magnitude, argument and distance in complex plane
- Rotations, colinearity, perpendicularity

Sample Problem:
(Math Day at the Beach-2010-Team-6) Let $z_1, z_2, \ldots, z_{10}$ be complex numbers that form a regular decagon (10-sided polygon) in the complex plane, with that decagon inscribed in a circle of radius $\sqrt[5]{7}$ centered at 2. At least one of the z_k is real. Compute the product $z_1 z_2 \cdots z_{10}$.

Chapter 12: 3D

- Platonic solids, spheres, cylinders, cones
- Distance formula, point-to-plane formula, Euler characteristic
- Using cross-sections and 2D properties to solve 3D problems

Sample Problem:
(AIME-2000-I-8) A container in the shape of a right circular cone is 12 inches tall and its base has a 5-inch radius. The liquid that is sealed inside is 9 inches deep

when the cone is held with its point down and its base horizontal. When the cone is held with its point up and its base horizontal, the height of the liquid is $m - n\sqrt[3]{p}$, where m, n, and p are positive integers and p is not divisible by the cube of any prime number. Find $m + n + p$.

MC40N
AIME Basic Number Theory

Chapter 1: Number Bases

- Non-decimal bases
- Legendre's formula

Sample Problem:
(Folklore) What is the 200^{th} smallest positive integer that can be written as the sum of distinct powers of 3?

Chapter 2: Primes & Prime Factorization

- Definitions of primes and Euclid's Lemma
- Fundamental Theorem of Arithmetic

Sample Problem:
(AIME-2006-I-4) Let N be the number of consecutive 0's at the right end of the decimal representation of the product $1!2!3!4!\dots 99!100!$. Find the remainder when N is divided by 1000.

Chapter 3: Divisibility Rules

- Divisibility rules
- p-adic valuation

- Lifting the exponent

Sample Problem:
(Brian Shimanuki) Construct the smallest positive integer divisible by 18 using only the digits 3 and 4.

Chapter 4: Multiplicative Functions

- Problems involving multiplicative functions, such as Divisor function, Sigma function, Totient function
- Properties of φ function

Sample Problem:
(BMT-2012-Tournament-Round7-P3) Let φ be the Euler totient function, and let $S = \{x \mid \frac{x}{\varphi(x)} = 3\}$. What is $\sum_{x \in S} \frac{1}{x}$?

Chapter 5: Factoring Techniques

- Difference of squares and arbitrary powers, sum of cubes and odd powers
- Simon's Favorite Factoring Trick
- Sophie Germain identity

Sample Problem:
(HMMT Nov-2008-Guts-31) Find the sum of all primes p for which there exists a prime q such that $p^2 + pq + q^2$ is a square.

Chapter 6: GCD & LCM

- Greatest common divisor, least common multiple
- Euclidean algorithm and its applications
- Bezout's identity

Sample Problem:
(Hong Kong MC-2006-18) For any positive integer n, let $f(n) = 70 + n^2$ and $g(n)$ be the H.C.F. of $f(n)$ and $f(n+1)$. Find the greatest possible value of $g(n)$.

Chapter 7: Modular Arithmetic

- Properties of modulo
- Modular inverses
- Using binomial theorem to find remainders
- Solving AIME level problems using modular arithmetic

Sample Problem:
(AMC12-2014-B23) The number 2017 is prime. Let $S = \sum_{k=0}^{62} \binom{2014}{k}$. What is the remainder when S is divided by 2017?

(A) 32 **(B)** 684 **(C)** 1024 **(D)** 1576 **(E)** 2016

Chapter 8: Fermat's Little Theorem & Euler Theorem

- Fermat's little theorem
- Euler's totient theorem
- Wilson's theorem

Sample Problem:
(ARML-2002-Individual-6) Let a be the integer such that $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots+\frac{1}{22}+\frac{1}{23}=\frac{a}{23!}$. Compute the remainder when a is divided by 13.

Chapter 9: Chinese Remainder Theorem

- Chinese remainder theorem (CRT)
- Computing solutions to CRT, using CRT backwards

Sample Problem:
(PUMaC-2008-Number Theory-5) If $f(x) = x^{x^{x^x}}$, find the last two digits of $f(17) + f(18) + f(19) + f(20)$.

Chapter 10: Degree

- Definition and properties of order modulo m

Sample Problem:
(CHMMC-2014-Individual-7) A robot is shuffling a 9 card deck. Being very well machined, it does every shuffle in exactly the same way: it splits the deck into two piles, one containing the 5 cards from the bottom of the deck and the other with the 4 cards from the top. It then interleaves the cards from the two piles, starting with a card from the bottom of the larger pile at the bottom of the new deck, and then alternating cards from the two piles while maintaining the relative order of each pile. The top card of the new deck will be the top card of the bottom pile.
The robot repeats this shuffling procedure a total of n times, and notices that the cards are in the same order as they were when it started shuffling. What is the smallest possible value of n?

Chapter 11: Primitive Roots

- Definition of primitive roots
- Primitive root theorem
- Finding number of equations of modular equations using primitive roots

Sample Problem:
(Kevin Li) How many primitive roots does 1458 have?

Chapter 12: Diophantine Equations

- General strategies for solving AIME level diophantine equations
- Chicken McNugget Theorem
- Computing Pythagorean triples
- Fermat's Last Theorem (optional)

Sample Problem:
(PUMaC-2014-Number Theory-5) Find the number of pairs of integer solutions (x, y) that satisfies the equation

$$(x-y+2)(x-y-2) = -(x-2)(y-2).$$

MC45A
AIME Advanced Algebra

Chapter 1: Word Problems

- Developing logical analysis and boost creative thinking by solving word problems.
- Converting word problems into mathematical equations and solving AIME level system of equations.

Sample Problem:
(HMMT Feb-2012-Guts-17) Mark and William are playing a game. Two walls are placed 1 meter apart, with Mark and William each starting an orb at one of the walls. Simultaneously, they release their orbs directly toward the other. Both orbs are enchanted such that, upon colliding with each other, they instantly reverse direction and go at double their previous speed. Furthermore, Mark has enchanted his orb so that when it collides with a wall it instantly reverses direction and goes at double its previous speed (William's reverses direction at the same speed). Initially, Mark's orb is moving at 1/1000 meters/s, and William's orb is moving at 1 meter/s. Mark wins when his orb passes the halfway point between the two walls. How fast, in meters/s, is his orb going when this first happens?

Chapter 2: Sequences & Series

- Finding patterns in sequences by looking at small cases.
- Using trig substitution and invariance in sequence problems.
- Understanding recurrence relations and solving linear recurrences.

- Finding closed-form formulas for sequences.

Sample Problem:
(AIME-2008-I-12) On a long straight stretch of one-way single-lane highway, cars all travel at the same speed and all obey the safety rule: the distance from the back of the car ahead to the front of the car behind is exactly one car length for each 15 kilometers per hour of speed or fraction thereof (Thus the front of a car traveling 52 kilometers per hour will be four car lengths behind the back of the car in front of it.) A photoelectric eye by the side of the road counts the number of cars that pass in one hour. Assuming that each car is 4 meters long and that the cars can travel at any speed, let M be the maximum whole number of cars that can pass the photoelectric eye in one hour. Find the quotient when M is divided by 10.

Chapter 3: Functions-1

- Solving equations that involve special functions such as floor, ceiling and absolute value
- Counting functions using information about its domain and range

Sample Problem:
(Mohammad Jafari) Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(f(x + f(y))) = 2x + f(x + y)$$

for all real x, y.

Chapter 4: Functions-2

- Solving functional equations using substitution, injectivity, and surjectivity, symmetry

Sample Problem:
(HMMT Nov-2015-Guts-26) Let $f : \mathbb{R}^+ \to \mathbb{R}$ be a *continuous* function satisfying $f(xy) = f(x) + f(y) + 1$ for all positive reals x, y. If $f(2) = 0$, compute $f(2015)$.

Chapter 5: Polynomials-1

- Finding roots of some cubic, quartic, and higher degree polynomials using substitution, binomial theorem
- Vieta's theorem and its applications
- Using techniques such as long division, factor theorem and rational root theorem when finding roots of higher degree polynomials

Sample Problem:
(PUMaC-2010-Algebra-5) Let $f(x) = 3x^3 - 5x^2 + 2x - 6$. If the roots of f are given by α, β, and γ, find

$$\left(\frac{1}{\alpha - 2}\right)^2 + \left(\frac{1}{\beta - 2}\right)^2 + \left(\frac{1}{\gamma - 2}\right)^2.$$

Chapter 6: Polynomials-2

- Solving polynomial equations using Lagrange interpolation and Finite differences

Sample Problem:
(HMMT Feb-2010-Algebra-6) Suppose that a polynomial of the form $p(x) = x^{2010} \pm x^{2009} \pm \cdots \pm x \pm 1$ has no real roots. What is the maximum possible number of coefficients of -1 in p.

Chapter 7: Logarithm

- Solving AIME level problems involving logarithms and natural logarithm

Sample Problem:
(AIME-2006-I-9) The sequence $a_1, a_2, \ldots$ is geometric with $a_1 = a$ and common ratio r, where a and r are positive integers. Given that $\log_8 a_1 + \log_8 a_2 + \cdots + \log_8 a_{12} = 2006$, find the number of possible ordered pairs (a, r).

Chapter 8: Trigonometry

- Solving algebra problems using trig substitution, trig identities and formulas

Sample Problem:
(SMT-2014-Algebra Tiebreaker-3) Compute $\dfrac{1}{\sin^2 \frac{\pi}{10}} + \dfrac{1}{\sin^2 \frac{3\pi}{10}}$.

Chapter 9: Complex Numbers-1

- Having a deep knowledge of complex numbers, finding roots of polynomials with complex roots
- Algebraic operations involving complex numbers and complex plane
- Problem solving techniques using Euler's formula and de Moivre's formula

Sample Problem:
(AIME-2013-I-14) For $\pi \le \theta < 2\pi$, let

$$P = \frac{1}{2}\cos\theta - \frac{1}{4}\sin 2\theta - \frac{1}{8}\cos 3\theta + \frac{1}{16}\sin 4\theta + \frac{1}{32}\cos 5\theta - \frac{1}{64}\sin 6\theta - \frac{1}{128}\cos 7\theta + \dots$$

and

$$Q = 1 - \frac{1}{2}\sin\theta - \frac{1}{4}\cos 2\theta + \frac{1}{8}\sin 3\theta + \frac{1}{16}\cos 4\theta - \frac{1}{32}\sin 5\theta - \frac{1}{64}\cos 6\theta + \frac{1}{128}\sin 7\theta + \dots$$

so that $\frac{P}{Q} = \frac{2\sqrt{2}}{7}$. Then $\sin\theta = -\frac{m}{n}$ where m and n are relatively prime positive integers. Find $m + n$.

Chapter 10: Complex Numbers-2

- Finding roots of unity and using algebraic operations on roots of unity to solve problems

Sample Problem:
(HMMT Feb-2006-Guts-21) Find the smallest positive integer k such that $z^{10} + z^9 + z^6 + z^5 + z^4 + z + 1$ divides $z^k - 1$.

Chapter 11: System of Equations

- Solving system of equations using polynomials, substitutions and symmetry

Sample Problem:
(PUMaC-2016-Combinatorics-3) Alice, Bob, Charlie, Diana, Emma, and Fred sit in a circle, in that order, and each roll a six-sided die. Each person looks at his or her own roll, and also looks at the roll of either the person to the right or to the left, deciding at random. Then, at the same time, Alice, Bob, Charlie, Diana, Emma and Fred each state the expected sum of the dice rolls based on the information they have. All six people say different numbers; in particular, Alice, Bob, Charlie, and Diana say 19, 22, 21, and 23, respectively. Compute the product of the dice rolls.

Chapter 12: Inequalities

- Finding minimum/maximum of algebraic expressions using elementary properties of inequalities, such as transitivity and algebraic operations on inequalities
- Arithmetic Mean - Geometric Mean (AM-GM) Inequality
- Cauchy-Schwarz Inequality
- Some advanced inequalities such as Rearrangement Inequality, Jensen's Inequality and weighted AM-GM Inequality

Sample Problem:
(SMT-2018-Team-8) Eddy has two blank cubes A and B and a marker. Eddy is allowed to draw a total of 36 dots on cubes A and B to turn them into dice, where each side has an equal probability of appearing, and each side has at least one dot on it. Eddy then rolls dice A twice and dice B once and computes the product of the three numbers. Given that Eddy draws dots on the two dice to maximize his expected product, what is his expected product?

MC45C
AIME Advanced Counting

Chapter 1: Basic Counting Techniques

- Solving counting problems using techniques such as casework and complementary counting

Sample Problem:
(AIME-2016-I-8) For a permutation $p = (a_1, a_2, \ldots, a_9)$ of the digits $1, 2, \ldots, 9$, let $s(p)$ denote the sum of the three 3-digit numbers $a_1a_2a_3$, $a_4a_5a_6$, and $a_7a_8a_9$. Let m be the minimum value of $s(p)$ subject to the condition that the units digit of $s(p)$ is 0. Let n denote the number of permutations p with $s(p) = m$. Find $|m - n|$.

Chapter 2: Counting Sets & PIE

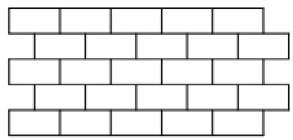
- Solving counting problems using the Principle of Inclusion and Exclusion (PIE)

Sample Problem:
(HMMT Feb-2006-Combinatorics-1) Vernonia High School has 85 seniors, each of whom plays on at least one of the school's three varsity sports teams: football, baseball, and lacrosse. It so happens that 74 are on the football team; 26 are on the baseball team; 17 are on both the football and lacrosse teams; 18 are on both the baseball and football teams; and 13 are on both the baseball and lacrosse teams. Compute the number of seniors playing all three sports, given that twice this number are members of the lacrosse team.

Chapter 3: Path Counting & Bijections

- Solving counting problems using bijections
- Solving path-counting problems

Sample Problem:
(HMMT Feb-2012-Combinatorics-3) In the figure below, how many ways are there to select 5 bricks, one in each row, such that any two bricks in adjacent rows are adjacent?



Chapter 4: Stars and Bars

- Solving counting problems using the Stars and Bars method

Sample Problem:
(PUMaC-2014-Team-5) How many sets of positive integers (a, b, c) satisfies $a > b > c > 0$ and $a + b + c = 103$?

Chapter 5: Binomial

- Solving counting problems involving binomials and multinomials
- Binomial identities such as Hockey-Stick Identity and Vandermonde's Identity

Sample Problem:
(PUMaC-2008-Combinatorics-4) Find the sum of the values of x for which

$$\binom{x}{0} - \binom{x}{1} + \binom{x}{2} - \cdots + \binom{x}{2008} = 0.$$

Chapter 6: Counting with Recursion

- Identifying which counting problems can be solved using recursions
- Finding and solving recursions

Sample Problem:
(HMMT Feb-2002-Guts-13) A *domino* is a 1-by-2 or 2-by-1 rectangle. A *domino tiling* of a region of the plane is a way of covering it (and only it) completely by non-overlapping dominoes. For instance, there is one domino tiling of a 2-by-1 rectangle and there are 2 tilings of a 2-by-2 rectangle (one consisting of two horizontal dominoes and one consisting of two vertical dominoes). How many domino tilings are there of a 2-by-10 rectangle?

Chapter 7: Probability

- Solving difficult probability problems
- Conditional probability and Bayes' Theorem
- Geometric probability

Sample Problem:
(HMMT Feb-2010-Guts-16) Jessica has three marbles colored red, green, and blue. She randomly selects a non-empty subset of them (such that each subset is equally likely) and puts them in a bag. You then draw three marbles from the bag with replacement. The colors you see are red, blue, red. What is the probability that the only marbles in the bag are red and blue?

Chapter 8: Expected Value

- Random variables, expected value and variance
- Solving geometry problems involving expected values
- Properties of expectation, such as linearity of expectation

Sample Problem:
(BMT-2012-Tournament-Round2-P1) 4 balls are distributed uniformly at random among 6 bins. What is the expected number of empty bins?

Chapter 9: Markov Chains

- Solving problems using Markov chains and state diagrams

Sample Problem:
(PUMaC-2010-Combinatorics-4) Erick stands in the square in the 2nd row and 2nd column of a 5 by 5 chessboard. There are \$1 bills in the top left and bottom right squares and there are \$5 bills in the top right and bottom left squares, as shown below.

\$1				\$5
	E			
\$5				\$1

Every second, Erick randomly chooses a square adjacent to the one he currently stands in (that is, a square sharing an edge with the one he currently stands in) and moves to that square. When Erick reaches a square with money on it, he takes it and quits. The expected value of Erick's winnings in dollars is m/n, where m and n are relatively prime positive integers. Find $m + n$.

Chapter 10: Geometric Counting

- Solving counting problems related to geometric objects
- Euler's Formula

Sample Problem:
(AIME-2018-I-7) A right hexagonal prism has height 2. The bases are regular hexagons with side length 1. Any 3 of the 12 vertices determine a triangle. Find the number of these triangles that are isosceles (including equilateral triangles).

Chapter 11: Generating Functions

- Using generating functions to turn counting problems into algebra
- Counting number of partitions

Sample Problem:
(CHMMC-2010 Winter-Team-6) Zach rolls five tetrahedral dice, each of whose faces are labeled 1, 2, 3, and 4. Compute the probability that the sum of the values of the faces that the dice land on is divisible by 3.

Chapter 12: Catalan Numbers

- Using Catalan numbers to solve counting problems

Sample Problem:
(HMMT Feb-2007-Combinatorics-10) A subset S of the nonnegative integers is called *supported* if it contains 0, and $k+8, k+9 \in S$ for all $k \in S$. How many supported sets are there?

MC45G
AIME Advanced Geometry

Chapter 1: Angles

- Angles in circles and polygons; cyclic quadrilaterals
- Using angle chasing to solve problems

Sample Problem:
(CHMMC-2016-Individual-9) In quadrilateral $ABCD$, $AB = DB$ and $AD = BC$. If $m\angle ABD = 36^\circ$ and $m\angle BCD = 54^\circ$, find $m\angle ADC$ in degrees.

Chapter 2: Special Triangles

- equilateral, 30-60-90, 45-45-90, 15-75-90, 45-60-75, 36-72-72, and 18-72-90 triangles
- Pythagorean triples and Heronian scalenes

Sample Problem: (Prasolov1 p101 q5.24) Points D and E divide sides AC and AB of an equilateral triangle ABC in the ratio of $AD : DC = BE : EA = 1 : 2$. Lines BD and CE meet at point O. Prove that $\angle AOC = 90^\circ$.

Chapter 3: Similarity

- Similarity and congruence conditions (SSS, ASA, SAA, AA and SAS similarity, not SSA)
- Angle bisector theorem

Sample Problem:
(PUMaC-2016-Geometry-7) Let $ABCD$ be a cyclic quadrilateral with circumcircle ω and let AC and BD intersect at X. Let the line through A parallel to BD intersect line CD at E and ω at $Y \neq A$. If $AB = 10$, $AD = 24$, $XA = 17$, and $XB = 21$, then the area of ΔDEY can be written in simplest form as $\frac{m}{n}$. Find $m + n$.

Chapter 4: Special Points

- Properties of the four triangle centers (centroid, orthocenter, incenter, circumcenter)
- The Euler line

Sample Problem:
(AIME-2010-I-15) In $\triangle ABC$ with $AB = 12$, $BC = 13$, and $AC = 15$, let M be a point on $\overline{AC}$ such that the incircles of $\triangle ABM$ and $\triangle BCM$ have equal radii. Let p and q be positive relatively prime integers such that $\frac{AM}{CM} = \frac{p}{q}$. Find $p + q$.

Chapter 5: Length-1

- Triangle inequality and Ravi substitution
- Pythagorean theorem and distance formula
- Mass points
- Ceva's Theorem, Menelaus' Theorem and Stewart's Theorem

Sample Problem:

Chapter 6: Length-2

- Length problems involving circles
- Power of a point
- Radical axis and radical center
- Ptolemy's theorem

Sample Problem:
(PUMaC-2014-Geometry-4) Consider the cyclic quadrilateral with sides 1, 4, 8, 7 in that order. What is its circumdiameter? Let the answer be of the form $a\sqrt{b}+c$, for b square free. Find $a+b+c$.

Chapter 7: Area-1

- Triangle area formulas
- Special quadrilateral area formulas such as Brahmagupta's formula

Sample Problem:
(AIME-2018-I-12) For each subset T of $U = \{1, 2, 3, \cdots, 18\}$, let $s(T)$ be the sum of the elements of T, with $s(\varnothing)$ defined to be 0. If T is chosen at random among all subsets of U, the probability that $s(T)$ is divisible by 3 is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find m.

Chapter 8: Area-2

- Area problems involving length ratios

Sample Problem:
(HMMT Feb-2004-Geometry-10) Right triangle XYZ has right angle at Y and $XY = 228$, $YZ = 2004$. Angle Y is trisected, and the angle trisectors intersect XZ at P and Q so that X, P, Q, Z lie on XZ in that order. Find the value of $(PY+YZ)(QY+XY)$.

Chapter 9: Trigonometry

- Definitions of trigonometric functions, basic trig identities, sum and difference formulas
- Law of sines, law of cosines, ratio lemma
- Trigonometric Ceva

Sample Problem:
(AIME-2018-I-13) Let ΔABC have side lengths $AB = 30$, $BC = 32$, and $AC = 34$. Point X lies in the interior of $\overline{BC}$, and points I_1 and I_2 are the incenters of ΔABX and ΔACX, respectively. Find the minimum possible area of ΔAI_1I_2 as X varies along BC.

Chapter 10: Analytic Geometry

- Distance formulas (between two points, point & line)
- The slope and the equation of a line (slope-intercept and point slope)
- Reflections over lines
- Equation of circles
- Shoelace formula and Pick's theorem

Sample Problem:
(HMMT Feb-2010-Guts-13) A triangle in the xy-plane is such that when projected onto the x-axis, y-axis, and the line $y = x$, the results are line segments whose endpoints are $(1, 0)$ and $(5, 0)$, $(0, 8)$ and $(0, 13)$, and $(5, 5)$ and $(7.5, 7.5)$, respectively. What is the triangle's area?

Chapter 11: Complex Numbers

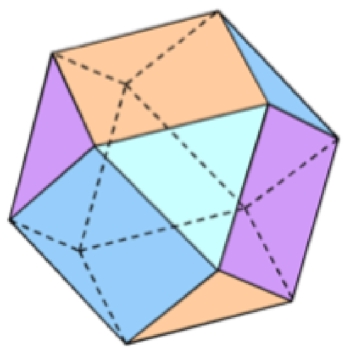
- Introduction to radians, Euler's formula
- Various representations of complex numbers
- Magnitude, argument and distance in complex plane
- Rotations, colinearity, perpendicularity

Sample Problem:
(AIME-2012-I-14) Complex numbers a, b, and c are zeros of a polynomial $P(z) = z^3 + qz + r$, and $|a|^2 + |b|^2 + |c|^2 = 250$. The points corresponding to a, b, and c in the complex plane are the vertices of a right triangle with hypotenuse h. Find h^2.

Chapter 12: 3D

- Platonic solids, spheres, cylinders, cones
- Distance formula, point-to-plane formula, Euler characteristic
- Using cross-sections and 2D properties to solve 3D problems

Sample Problem:
(PUMaC-2010-Geometry-5) A cuboctahedron is a solid with 6 square faces and 8 equilateral triangle faces, with each edge adjacent to both a square and a triangle (see picture). Suppose the ratio of the volume of an octahedron to a cuboctahedron with the same side length is r. Find $100r^2$.



MC45N
AIME Advanced Number Theory

Chapter 1: Number Bases

- Non-decimal bases
- Legendre's formula

Sample Problem:
(AIME-2010-I-10) Let N be the number of ways to write 2010 in the form

$$2010 = a_3 \cdot 10^3 + a_2 \cdot 10^2 + a_1 \cdot 10 + a_0,$$

where the a_i's are integers, and $0 \le a_i \le 99$. An example of such a representation is $1 \cdot 10^3 + 3 \cdot 10^2 + 67 \cdot 10^1 + 40 \cdot 10^0$. Find N.

Chapter 2: Primes & Prime Factorization

- Definitions of primes and Euclid's Lemma
- Fundamental Theorem of Arithmetic

Sample Problem: (1001 Problems in NT p26 q153) Find the smallest positive integer n such that $n/2$ is a perfect square, $n/3$ is a cube and $n/5$ is a fifth power.

Chapter 3: Divisibility Rules

- Divisibility rules

- p-adic valuation
- Lifting the exponent

Sample Problem:
(AIME-2006-II-14) Let S_n be the sum of the reciprocals of the nonzero digits of the integers from 1 to 10^n, inclusive. Find the smallest positive integer n for which S_n is an integer.

Chapter 4: Multiplicative Functions

- Problems involving multiplicative functions, such as Divisor function, Sigma function, Totient function
- Properties of φ function

Sample Problem:
(AIME-2016-II-11) For positive integers N and k, define N to be k-nice if there exists a positive integer a such that a^k has exactly N positive divisors. Find the number of positive integers less than 1000 that are neither 7-nice nor 8-nice.

Chapter 5: Factoring Techniques

- Difference of squares and arbitrary powers, sum of cubes and odd powers
- Simon's Favorite Factoring Trick
- Sophie Germain identity

Sample Problem:
(CHMMC-2012 Fall-Individual-8) Find two pairs of positive integers (a, b) with $a > b$ such that

$$a^2 + b^2 = 40501.$$

Chapter 6: GCD & LCM

- Greatest common divisor, least common multiple
- Euclidean algorithm and its applications

- Bezout's identity

Sample Problem:
(CHMMC-2010 Fall-Team-5) The three positive integers a, b, c satisfy the equalities $\gcd(ab, c^2) = 20$, $\gcd(ac, b^2) = 18$, and $\gcd(bc, a^2) = 75$. Compute the minimum possible value of $a + b + c$.

Chapter 7: Modular Arithmetic

- Properties of modulo
- Modular inverses
- Using binomial theorem to find remainders
- Solving AIME level problems using modular arithmetic

Sample Problem:
(AIME-2012-I-15) There are n mathematicians seated around a circular table with n seats numbered 1, 2, 3, ..., n in clockwise order. After a break they again sit around the table. The mathematicians note that there is a positive integer a such that

1. for each k, the mathematician who was seated in seat k before the break is seated in seat ka after the break (where seat $i + n$ is seat i);
2. for every pair of mathematicians, the number of mathematicians sitting between them after the break, counting in both the clockwise and the counter-clockwise directions, is different from either of the number of mathematicians sitting between them before the break.

Find the number of possible values of n with $1 < n < 1000$.

Chapter 8: Fermat's Little Theorem & Euler Theorem

- Fermat's little theorem
- Euler's totient theorem
- Wilson's theorem

Sample Problem:
(HMMT Feb-2010-Guts-29) Compute the remainder when

$$\sum_{k=1}^{30303} k^k$$

is divided by 101.

Chapter 9: Chinese Remainder Theorem

- Chinese remainder theorem (CRT)
- Computing solutions to CRT, using CRT backwards

Sample Problem:
(AIME-2012-I-10) Let $\mathcal{S}$ be the set of all perfect squares whose rightmost three digits in base 10 are 256. Let $\mathcal{T}$ be the set of all numbers of the form $\frac{x-256}{1000}$, where x is in $\mathcal{S}$. In other words, $\mathcal{T}$ is the set of numbers that result when the last three digits of each number in $\mathcal{S}$ are truncated. Find the remainder when the tenth smallest element of $\mathcal{T}$ is divided by 1000.

Chapter 10: Degree

- Definition and properties of order modulo m

Sample Problem:
(AIME-2018-I-11) Find the least positive integer n such that when 3^n is written in base 143, its two right-most-digits in base 143 are 01.

Chapter 11: Primitive Roots

- Definition of primitive roots
- Primitive root theorem
- Finding number of equations of modular equations using primitive roots

Sample Problem:
(CHMMC-2010 Winter-Individual-15) Compute the number of primes p less than 100 such that p divides $n^2 + n + 1$ for some integer n.

Chapter 12: Diophantine Equations

- General strategies for solving AIME level diophantine equations
- Chicken McNugget Theorem
- Computing Pythagorean triples
- Fermat's Last Theorem (optional)

Sample Problem:
(AIME-2008-II-15) Find the largest integer n satisfying the following conditions:
(i) n^2 can be expressed as the difference of two consecutive cubes;
(ii) $2n + 79$ is a perfect square.